11-1 Skills Practice

Arithmetic Sequences

Find the next four terms of each arithmetic sequence.

1. 7, 11, 15, ... 19, 23, 27, 31
2. −10, −5, 0, ... 5, 10, 15, 20
3. 101, 202, 303, ... 404, 505, 606, 707
4. 15, 7, −1, ... −9, −17, −25, −33
5. −67, −60, −53, ...
   −64, −68, −72, −76
6. −12, −15, −18, ...
   −21, −24, −27, −30

Find the first five terms of each arithmetic sequence described.

7. \( a_1 = 6, \ d = 9 \) 6, 15, 24, 33, 42
8. \( a_1 = 27, \ d = 4 \) 27, 31, 35, 39, 43
9. \( a_1 = −12, \ d = 5 \) −12, −7, −2, 3, 8
10. \( a_1 = 93, \ d = −15 \) 93, 78, 63, 48, 33
11. \( a_1 = −64, \ d = 11 \) −64, −53, −42, −31, −20
12. \( a_1 = −47, \ d = −20 \) −47, −67, −87, −107, −127

Find the indicated term of each arithmetic sequence.

13. \( a_2 = 2, \ d = 6, \ n = 12 \) 68
14. \( a_1 = 18, \ d = 2, \ n = 8 \) 32
15. \( a_1 = 23, \ d = 5, \ n = 23 \) 133
16. \( a_1 = 15, \ d = −1, \ n = 25 \) −9
17. \( a_{31} \) for 34, 38, 42, ... 154
18. \( a_{42} \) for 27, 30, 33, ... 150

Complete the statement for each arithmetic sequence.

19. 55 is the \( \_\_\_\_ \) th term of 4, 7, 10, ...
20. 163 is the \( \_\_\_\_ \) th term of −5, 2, 9, ...

Write an equation for the \( n \)th term of each arithmetic sequence.

21. \( a_n = 3n + 1 \)
22. \( a_n = 2n + 3 \)
23. \( a_n = 4n − 5 \)
24. \( a_n = −5n + 12 \)

Find the arithmetic means in each sequence.

25. 6, 7, 8, 9, 14, 22, 30
26. 63, 66, 69, 72, 147, 84, 105, 126

11-1 Practice

Arithmetic Sequences

Find the next four terms of each arithmetic sequence.

1. 5, 8, 11, ... 14, 17, 20, 23
2. −4, −6, −8, ... −10, −12, −14, −16
3. 100, 93, 86, ... 79, 72, 65, 58
4. −24, −19, −14, ... −9, −4, 1, 6
5. \( \frac{5}{2}, \frac{3}{2}, \frac{1}{2}, \ldots \) 7, 5, 3, 1, 21
6. 6, 4.4, 3.2, 2, 2.8, 2.6, 2.4, 2.2

Find the first five terms of each arithmetic sequence described.

7. \( a_1 = 7, \ d = 7 \)
8. \( a_1 = −8, \ d = 2 \)
9. 7, 14, 21, 28, 35
10. \( a_1 = 9, \ d = −4 \)
11. \( a_1 = −7, \ d = −2 \)
12. \( a_1 = 10, \ d = −5, 8 \)
13. \( a_1 = −12, \ d = 4 \)
14. \( a_1 = −5, \ d = −2, 0 \)
15. \( a_1 = 12, \ d = −3, 4 \)
16. \( a_1 = 9, \ d = −3, 4 \)
17. \( a_1 = 5, \ d = −1, 2, 3 \)
18. \( a_1 = 15, \ d = −1, 2, 3 \)
19. \( a_1 = 10, \ d = −1, 2, 3 \)

Find the indicated term of each arithmetic sequence.

13. \( a_1 = 5, \ d = 3, \ n = 10 \) 32
14. \( a_1 = 9, \ d = 3, \ n = 32 \)
15. \( a_{30} \) for 124, 119, 114, ...
16. \( a_{30} \) for 5, 10, 15, ...
17. \( a_1 = 5, \ d = −3, \ n = 10 \) 18
18. \( a_1 = 14.25, \ d = 0.15, \ n = 31 \)

Complete the statement for each arithmetic sequence.

19. 166 is the \( \_\_\_\_ \) th term of 30, 34, 38, ...
20. 25 is the \( \_\_\_\_ \) th term of \( \frac{5}{2}, \frac{3}{2}, \frac{1}{2}, \ldots \)

Write an equation for the \( n \)th term of each arithmetic sequence.

21. \( a_n = 2n + 7 \)
22. \( a_n = 3n − 9 \)
23. \( a_n = 4n + 5 \)
24. \( a_n = −5n + 12 \)

Find the arithmetic means in each sequence.

25. \( a_n = 2n + 3 \)
26. \( a_n = 3n − 9 \)
27. \( a_n = 4n + 5 \)

28. \( a_n = −5n + 12 \)

29. \( a_n = 2n + 7 \)
30. \( a_n = 3n − 9 \)
31. \( a_n = 4n + 5 \)
32. \( a_n = −5n + 12 \)
11-1 Word Problem Practice  

**Arithmetic Sequences**

1. **ALLOWANCES** Mark has saved $370 for a scooter and continues to save his weekly allowance of $10. Find the amount Mark will have saved after 7 weeks.  

   $440

2. **GRAPHS** A financial officer is making a graph of a company’s financial performance for the month. The vertical axis is labeled “Monthly Profit.” The values range from $500 to $8000. There is not enough space along the vertical axis to write all the numbers between $5000 and $8000, so the financial officer decides to write only 7 numbers, evenly spaced, starting at $5000 and ending at $8000. What should the numbers along the vertical axis be?  

   $5000, 6560, 7900, 6150, 6400, 6650, 6900$

3. **BIKING** City planners want to mark a bike trail with posts that give the distance along the trail to City Hall. The trail begins 37.2 miles from City Hall and ends at City Hall. Write a formula for the number of miles on the nth post if posts are placed every half mile starting at 37.2 miles and decreasing along the way to City Hall.  

   $37.7 - 0.5n$

4. **SEATING** Kay is trying to find her seat in a theater. The seats are numbered sequentially going left to right. Each row has 30 seats.  

   ![Seat Diagram]

   The figure shows some of the chairs in the left corner near the stage. Kay is at seat 129, but she needs to find seat 219. She notices that the seat numbers in a fixed column form an arithmetic sequence. What are the numbers of the next 4 seats in the same column as seat 129 going away from the stage? Where does Kay have to go to find her seat?  

   In what row and column is her seat?  

   159, 189, 219, 249; Kay can move 3 rows back; row 8, column 9

RINGS For Exercises 5-7, use the figure of expanding square rings.

5. How many small squares are in the first few square rings in the figure?  

   8, 16, 24

6. If the pattern continues, write a formula for the number of squares in the nth ring.  

   $8n$

7. What is the side length of the nth ring?  

   $2n + 1$

---

**Enrichment**

**Fibonacci Sequence**

Leonardo Fibonacci first discovered the sequence of numbers named for him while studying rabbits. He wanted to know how many pairs of rabbits would be produced in n months, starting with a single pair of newborn rabbits. He made the following assumptions.

1. Newborn rabbits become adults in one month.  
2. Each pair of rabbits produces one pair each month.  
3. No rabbits die.

Let $F_n$ represent the number of pairs of rabbits at the end of $n$ months. If you begin with one pair of newborn rabbits, $F_1 = F_0 = 1$. This pair of rabbits would produce one pair at the end of the second month, so $F_2 = 1 + 1 = 2$. At the end of the third month, the first pair of rabbits would produce another pair. Thus, $F_3 = 2 + 1 = 3$.

The chart below shows the number of rabbits each month for several months.

<table>
<thead>
<tr>
<th>Month</th>
<th>Adult Pairs</th>
<th>Newborn Pairs</th>
<th>Total</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>0</td>
<td>0</td>
<td>1</td>
</tr>
<tr>
<td>1</td>
<td>1</td>
<td>0</td>
<td>1</td>
</tr>
<tr>
<td>2</td>
<td>1</td>
<td>1</td>
<td>2</td>
</tr>
<tr>
<td>3</td>
<td>2</td>
<td>1</td>
<td>3</td>
</tr>
<tr>
<td>4</td>
<td>3</td>
<td>2</td>
<td>5</td>
</tr>
<tr>
<td>5</td>
<td>5</td>
<td>3</td>
<td>8</td>
</tr>
</tbody>
</table>

**Exercises**

Solve.

1. Starting with a single pair of newborn rabbits, how many pairs of rabbits would there be at the end of 12 months?  

   233

2. Write the first 10 terms of the sequence for which $F_0 = 3$, $F_1 = 4$, and $F_n = F_{n-1} + F_{n-2}$.  

   3, 4, 7, 11, 18, 29, 47, 76, 123, 199, 322

3. Write the first 10 terms of the sequence for which $F_0 = 1$, $F_1 = 1$, and $F_n = F_{n-1} + F_{n-2}$.  

   1, 5, 6, 11, 17, 28, 45, 73, 118, 191, 309
Lesson Reading Guide

Arithmetic Series

Get Ready for the Lesson

Read the introduction to Lesson 11-2 in your textbook.

Suppose that an amphitheater can seat 50 people in the first row and that each row thereafter can seat 9 more people than the previous row. Using the vocabulary of arithmetic sequences, describe how you would find the number of people who could be seated in the first 10 rows. (Do not actually calculate the sum.) Sample answer: Find the first 10 terms of an arithmetic sequence with first term 50 and common difference 9. Then add these 10 terms.

Read the Lesson

1. What is the relationship between an arithmetic sequence and the corresponding arithmetic series? Sample answer: An arithmetic sequence is a list of terms with a common difference between successive terms. The corresponding arithmetic series is the sum of the terms of the sequence.

2. Consider the formula \( S_n = \frac{n}{2}(a_1 + a_n) \). Explain the meaning of this formula in words.
   Sample answer: To find the sum of the first \( n \) terms of an arithmetic sequence, find half the number of terms you are adding. Multiply this number by the sum of the first term and the \( n \)th term.

3. a. What is the purpose of sigma notation?
   Sample answer: to write a series in a concise form

   b. Consider the expression \( \sum_{i=1}^{12} (4i - 2) \).
       This form of writing a sum is called _______.
       The variable \( i \) is called the _______.
       The first value of \( i \) is _______.
       The last value of \( i \) is _______.
       How would you read this expression? The sum of \( 4i - 2 \) as \( i \) goes from 2 to 12.

Remember What You Learned

4. A good way to remember something is to relate it to something you already know. How can your knowledge of how to find the average of two numbers help you remember the formula \( S_n = \frac{n}{2}(a_1 + a_n) \)? Sample answer: Rewrite the formula as
   \( S_n = \frac{n}{2}(a_1 + a_n) \). The average of the first and last terms is given by the expression \( \frac{a_1 + a_n}{2} \). The sum of the first \( n \) terms is the average of the first and last terms multiplied by the number of terms.

Arithmetic Series

An arithmetic series is the sum of consecutive terms of an arithmetic sequence.

\[
S_n = \frac{n}{2}(a_1 + a_n)
\]

**Example 1** Find \( S_{10} \) for the arithmetic series with \( a_1 = 14 \), \( a_n = 101 \), and \( n = 30 \).

Use the sum formula for an arithmetic series.

\[
S_{30} = \frac{30}{2}(14 + 101) = \frac{30}{2}(115) = 1725
\]

The sum of the series is 1725.

**Example 2** Find the sum of all positive odd integers less than 180.

The series is \( 1 + 3 + 5 + \ldots + 179 \).

Find \( n \) using the formula for the \( n \)th term of an arithmetic sequence.

\[
a_n = a_1 + (n - 1)d
\]

\[
a_{179} = a_1 + (179 - 1)d
\]

\[
179 = 1 + (179 - 1)d
\]

\[
178d = 178
\]

\[
d = 1
\]

Then use the sum formula for an arithmetic series.

\[
S_n = \frac{n}{2}(a_1 + a_n)
\]

\[
S_{180} = \frac{180}{2}(1 + 179) = 8100
\]

The sum of all positive odd integers less than 180 is 8100.

**Exercises**

Find \( S_n \) for each arithmetic series described.

1. \( a_1 = 12 \), \( a_n = 100 \), \( n = 12 \)

2. \( a_1 = 50 \), \( a_n = -50 \), \( n = 15 \)

3. \( a_1 = 60 \), \( a_n = -136 \), \( n = 50 \)

4. \( a_1 = 20 \), \( d = 4 \)

5. \( a_1 = -80 \), \( d = -8 \)

6. \( a_1 = -8 \), \( d = -7 \)

7. \( a_1 = 112 \)

8. \( a_1 = 1584 \)

9. \( a_1 = -71 \)

10. \( a_1 = -395 \)

11. \( a_1 = 504 \)

12. \( a_1 = 555 \)

13. \( a_1 = 1917 \)

**Find the sum of each arithmetic series.**

10. \( 8 + 6 + 4 + \ldots + 10 = 50 \)

11. \( 16 + 22 + 28 + \ldots + 112 = 1088 \)

**Find the first three terms of each arithmetic series described.**

12. \( a_1 = 12 \)

13. \( a_1 = 80 \)

14. \( a_1 = -115 \)

15. \( a_1 = 6.2 \), \( a_2 = 12.6 \)

\( S_3 = 1767 \)

16. \( a_1 = 80 \)

17. \( a_1 = 65 \)

18. \( a_1 = 50 \)

\( S_3 = -245 \)

19. \( a_1 = 84.6 \)

20. \( a_1 = 6.2 \), \( a_2 = 7.0 \)

\( S_3 = 7.0, 7.8 \)
11-2 Study Guide and Intervention (continued)

Arithmetic Series

Sigma Notation A shorthand notation for representing a series makes use of the Greek letter Σ. The sigma notation for the series $6 + 12 + 18 + 24 + 30$ is $\sum_{k=1}^{5} 6k$.

Example Evaluate $\sum_{k=1}^{15} (3k + 4)$.

The sum is an arithmetic series with common difference 3. Substituting $k = 1$ and $k = 15$ into the expression $3k + 4$ gives $a_1 = 3(1) + 4 = 7$ and $a_{15} = 3(15) + 4 = 58$. There are 18 terms in the series, so $n = 18$. Use the formula for the sum of an arithmetic series.

$$S_n = \frac{n}{2}(a_1 + a_n)$$

$$S_{18} = \frac{18}{2}(7 + 58)$$

$$= 9(65)$$

$$= 585$$

So $\sum_{k=1}^{15} (3k + 4) = 585$.

Exercises

Find the sum of each arithmetic series.

1. $\sum_{k=1}^{8} (2k + 1)$
2. $\sum_{k=1}^{8} (k - 1)$
3. $\sum_{k=1}^{14} (2k - 7)$
4. $\sum_{k=1}^{20} (2k - 200)$
5. $\sum_{k=1}^{10} (6k + 3)$
6. $\sum_{k=1}^{50} (500 - 6k)$
7. $\sum_{k=1}^{10} (100 - k)$
8. $\sum_{n=1}^{56} (n - 100)$
9. $\sum_{x=1}^{30} (2x)$
10. $\sum_{m=14}^{25} (2m - 50)$
11. $\sum_{p=1}^{20} (5p - 20)$
12. $\sum_{j=12}^{25} (25 - 2j)$
13. $\sum_{n=9}^{42} (4n - 9)$
14. $\sum_{x=4}^{50} (3n + 4)$
15. $\sum_{x=1}^{4}(7j - 3)$

Find the sum of each arithmetic series.

13. $4 + 7 + 10 + \ldots + 43$
14. $5 + 8 + 11 + 14 + \ldots + 32$
15. $3 + 5 + 7 + 9 + \ldots + 19$
16. $-2 + (-5) + (-8) + \ldots + (-20)$
17. $\sum_{k=1}^{15} (2k - 3)$
18. $\sum_{n=1}^{14} (10 + 3n)$
19. $\sum_{x=2}^{10} (4x + 1)$
20. $\sum_{n=3}^{12} (4 - 3n)$

Find the first three terms of each arithmetic series described.

21. $a_1 = 4, a_n = 31, S_n = 175$
22. $a_1 = -3, a_n = 41, S_n = 224$
23. $n = 10, a_n = 41, S_n = 230$
24. $n = 19, a_n = 85, S_n = 760$
1. Find $S_n$ for each arithmetic series described.

   1. $a_1 = 16, a_n = 98, n = 13$  
   2. $a_1 = 3, a_n = 36, n = 12$  
   3. $a_1 = -5, a_n = -26, n = 8$  
   4. $a_1 = 5, n = 10, a_n = -13$  
   5. $a_1 = 6, n = 15, a_n = -22$  
   6. $a_1 = -20, n = 25, a_n = 148$  
   7. $a_1 = 13, d = -6, n = 21$  
   8. $a_1 = 5, d = 4, n = 11$  
   9. $a_1 = 5, d = 2, n = 33$  
   10. $a_1 = -121, d = 3, n = 5$  
   11. $d = 0.4, n = 10, a_n = 3.8$  
   12. $d = -\frac{2}{3}, n = 16, a_n = 44$

2. Find the sum of each arithmetic series.

   13. $5 + 7 + 9 + 11 + \ldots + 27$  
   14. $-4 + 1 + 6 + 11 + \ldots + 91$  
   15. $13 + 20 + 27 + \ldots + 272$  
   16. $89 + 86 + 83 + 80 + \ldots + 20$  
   17. $\sum_{n=1}^{4} (2n - 1)$  
   18. $\sum_{n=1}^{3} (3n + 3n)$  
   19. $\sum_{n=1}^{3} (9 - 4n)$  
   20. $\sum_{n=1}^{3} (2n + 1)$  
   21. $\sum_{n=1}^{5} (5n - 10)$  
   22. $\sum_{n=1}^{10} (4 - 4n)$

3. Find the first three terms of each arithmetic series described.

   23. $a_1 = 14, a_n = -85, S_n = -1207$  
   24. $a_1 = 1, n = 19, S_n = 100$  
   25. $n = 16, a_n = 15, S_n = -120$  
   26. $n = 15, a_n = 54, S_n = 45$  
   27. $n = 30, a_n = 1, S_n = 30$  
   28. $n = 25, a_n = 5, S_n = 125$

4. VOLUNTEERING Maryland Public Schools requires all high school students to complete 75 hours of volunteer service as a condition for graduation. One school includes grades 1-12, with 50 students in each grade. The school decides that students in grade 9 will volunteer 0.25 hr per week of their time. How many hours will all the school's students collectively donate to charity each week?  975 hours

5. TRIANGLES For Exercises 5-7, use the following information.

   240

6. WEIGHTS Nathan has a collection of barbells for his home gym. He has 2 barbells for every 5 pounds starting at 5 pounds and going up to 80 pounds. What is the total weight of all his barbells? 1,360 lb

7. TRAINING Matthew is training to run a marathon. He runs 20 miles his first week of training. Each week, he increases the number of miles he runs by 4 miles. How many total miles did he run in 8 weeks of training? 272 mi

8. STACKING A health club rolls its towels and stacks them in layers on a shelf. Each layer of towels has one less towel than the layer below it. If there are 20 towels on the bottom layer and one towel on the top layer, how many towels are stacked on the shelf? 210 towels

9. BUSINESS A merchant places $1 in a jackpot on August 1, then draws the name of a regular customer. If the customer is present, he or she wins the $1 in the jackpot. If the customer is not present, the merchant adds $2 to the jackpot on August 2 and draws another name. Each day the merchant adds an amount equal to the day of the month. If the first person to win the jackpot wins $496, on what day of the month was her or his name drawn? August 31
11-2 Enrichment

Arithmetic Series in Computer Programming

Arithmetic series are used in the analysis of the efficiency of computer programs. Computers effortlessly automate time consuming, often repetitive tasks such as addition and multiplication of numbers. These repetitive tasks are carried out using a Loop statement provided by a programming language to execute the calculations until a logical condition is, or is not, satisfied. The loop usually repeats a calculation followed by an assignment statement, which is assigning the number to a specific memory location in the computer.

Suppose you were writing a program to calculate the sum of the numbers from 1 to 10, that is 1 + 2 + 3 + 4 + 5 + 6 + 7 + 8 + 9 + 10. Two algorithms to calculate this series are shown in the table with the sequential step of the algorithm in the left column.

<table>
<thead>
<tr>
<th>Step Number</th>
<th>Algorithm</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>Assign s = 1</td>
</tr>
<tr>
<td>2</td>
<td>Assign j = 2</td>
</tr>
<tr>
<td>3</td>
<td>If j &lt; 11 then do steps 4 and 5</td>
</tr>
<tr>
<td>4</td>
<td>Assign s = s + j</td>
</tr>
<tr>
<td>5</td>
<td>Assign j = j + 1</td>
</tr>
</tbody>
</table>

1. Write an algorithm segment in pseudo-code (like in the table) which for any given values of a, d, and n—the initial value, the common difference, and the number of terms in the progression, respectively—computes the sum of the series, $\sum_{i=1}^{n} a_i$.

   sum = a
   for i = 1 to n
   sum = sum + d
   next i

2. Double summations are used to analyze nested loops (loops inside of loops). Calculate the double summation below. Start with the inner summation first and then proceed to the outer summation.

   $\sum_{i=1}^{n} \sum_{j=1}^{m} (i + j + 3i) = \sum_{i=1}^{n} (i + 6i) = 6 + 12 + 18 + 24 = 60.

   Also recall the sum of a arithmetic series is equal to $\frac{n}{2}(a_1 + a_n)$, where $n$ is the number of terms in the series, $a_1$ is the first term of the sequence and $a_n$ is the last term.

   $a)\ \sum_{i=1}^{n} \sum_{j=1}^{m} (i + 3j) = 30 \quad b)\ \sum_{i=1}^{n} \sum_{j=1}^{m} \frac{n}{2} \times (n + 1) \times \left(\frac{m^2 + m}{2}\right)$

Chapter 11

Glencoe Algebra 2
Lesson 11-3

Lesson Reading Guide

Geometric Sequences

Get Ready for the Lesson

Read the introduction to Lesson 11-3 in your textbook.

Suppose that you drop a ball from a height of 4 feet, and that each time it falls, it bounces back to 74% of the height from which it fell. Describe how you would find the height of the third bounce. (Do not actually calculate the height of the bounce.)

Sample answer: Multiply 4 by 0.74 three times.

Read the Lesson

1. Explain the difference between an arithmetic sequence and a geometric sequence.

Sample answer: In an arithmetic sequence, each term after the first is found by adding the common difference to the previous term. In a geometric sequence, each term after the first is found by multiplying the previous term by the common ratio.

2. Consider the formula $a_n = a_1 \cdot r^{n-1}$.
   a. What is this formula used to find? A particular term of a geometric sequence.
   b. What do each of the following represent?
      
      $a_n$: the $n$th term
      
      $a_1$: the first term

      $r$: the common ratio

      $n$: a positive integer that indicates which term you are finding

3. a. In the sequence 5, 8, 11, 14, 17, 20, the numbers 8, 11, 14, and 17 are arithmetic means between 5 and 20.
   b. In the sequence 12, 4, $\frac{4}{3}$, $\frac{4}{9}$, $\frac{4}{27}$, the numbers $\frac{4}{3}$, $\frac{4}{9}$, and $\frac{4}{27}$ are geometric means between 12 and $\frac{4}{27}$.

Remember What You Learned

4. Suppose that your classmate Ricardo has trouble remembering the formula $a_n = a_1 \cdot r^{n-1}$ correctly. He thinks that the formula should be $a_n = a_1 \cdot r^n$. How would you explain to him that he should use $r^{n-1}$ rather than $r^n$ in the formula?

Sample answer: Each term after the first in a geometric sequence is found by multiplying the previous term by $r$. There are $n - 1$ terms before the $n$th term, so you would need to multiply by $r$ a total of $n - 1$ times, not $n$ times, to get the $n$th term.
11-3 Study Guide and Intervention (continued)

Geometric Sequences

Geometric Means: The geometric means of a geometric sequence are the terms between any two nonconsecutive terms of the sequence.

To find the k geometric means between two terms of a sequence, use the following steps.

Step 1: Let the two terms given be a1 and ak, where n = k + 1.
Step 2: Substitute in the formula ak = a1 r^k−1. a1 = a1, r = a1/a2.
Step 3: Solve for r, and use that value to find the k geometric means.

Example: Find the three geometric means between 8 and 40.5.

Use the nth term formula to find the value of r. In the sequence 8, _, _, _, _, 40.5, a1 is 8 and a5 is 40.5.

\[ a_5 = a_1 r^{5-1} \]
\[ 40.5 = 8 r^4 \]
\[ r^4 = 5.0625 \]
\[ r = \pm 1.5 \]

Take the fourth root of each side.

There are two possible common ratios, so there are two possible sets of geometric means.

Use each value of r to find the geometric means.

\[ r = 1.5 \]
\[ a_2 = 8(1.5) \text{ or } 12 \]
\[ a_3 = 8(1.5)^2 \text{ or } 18 \]
\[ a_4 = 8(1.5)^3 \text{ or } 27 \]

The geometric means are 12, 18, and 27, or -12, 18, and -27.

Exercises

Find the geometric means in each sequence.

1. 5, _, _, _, 405
   \[ \pm 15, 45, \pm 135 \]
2. 5, _, _, _, 20,48
   \[ 8, 12, 18 \]
3. 2, _, _, _, 375
   \[ \pm 3, 15, \pm 75 \]
4. _, 24, _, 1
   \[ 4, \frac{2}{3} \]
5. 12, _, _, _, 7,7,7
   \[ \pm 6, 3, \pm 3, \pm 3, \pm 3 \]
6. 200, _, _, _, 414.72
   \[ \pm 240, 288, \pm 345.6 \]
7. 24, _, _, _, 7,7,7,7,7
   \[ \pm 32, 35, \pm 12005 \]
8. 8, 7, 7, 7, 166.32
   \[ \pm 10, 25, \pm 62, \pm 2 \]
9. 9, _, _, _, _, _, _, 9
   \[ \pm 1, \pm 3, \pm 1, \pm 3 \]
10. 100, _, _, _, _, 384.16
    \[ \pm 140, 196, \pm 274.4 \]

Chapter 11  22  Glencoe Algebra 2

11-3 Skills Practice

Geometric Sequences

Find the next two terms of each geometric sequence.

1. -1, 2, -4, ..., -8, 16
   2. 6, 3, \frac{3}{2}, ..., \frac{3}{4}, \frac{3}{8}
3. -5, -15, -45, ..., -135, -405
   4. 729, -243, 81, ..., -27, 9
5. 1536, 384, 96, ..., 24, 6
   6. 64, 160, 400, ..., 1000, 2500

Find the first five terms of each geometric sequence described.

7. a1 = 6, r = 2
   8. a1 = -27, r = 3
   9. a1 = -15, r = -1
      10. a1 = 3, r = 4
      11. a1 = 1, r = \frac{1}{2}
      12. a1 = 216, r = \frac{1}{2}

Find the indicated term of each geometric sequence.

13. a1 = 5, r = 2, n = 6
14. a1 = 18, r = 3, n = 6
15. a1 = -3, r = -2, n = 5
16. a1 = -20, r = -2, n = 9
17. a4 for 1-, 6-, 3-, ..., \frac{3}{32}
18. a7 for 80, 80, 80, 80, 80, 80, ...

Write an equation for the nth term of each geometric sequence.

19. 3, 9, 27, ..., a_n = 3^n
20. -1, -3, -9, ..., a_n = -1(3)^n - 1
21. 2, -6, 18, ..., a_n = 2(-3)^n - 1
22. 5, 10, 20, ..., a_n = 5(2)^n - 1

Find the geometric means in each sequence.

23. 4, _, _, _, 64 ± 8, 16, ± 32
24. 1, _, _, _, 81 ± 3, 9, ± 27
11-3

Geometric Sequences

Find the next two terms of each geometric sequence.

1. $-15, -30, -60, \ldots -120, -240$
2. $80, 40, 20, \ldots 10, 5$

3. $30, 30, 30, \ldots \frac{10}{3}, \frac{10}{9}$
4. $-1458, 486, -162, \ldots 54, -18$

5. $\frac{1}{3}, \frac{1}{6}, \frac{1}{12}, \ldots 27, 81$
6. $216, 144, 96, \ldots 64, 128$

Find the first five terms of each geometric sequence described.

7. $a_1 = -1, r = 3$
8. $a_1 = 7, r = 4$
9. $a_1 = -\frac{1}{3}, r = 3$
10. $a_1 = 12, r = \frac{2}{3}$

Find the indicated term of each geometric sequence.

11. $a_5 = 5, r = 3, n = 6$
12. $a_1 = 20, r = -3, n = 6$

13. $a_2 = -4, r = -2, n = 10$
14. $a_6 \text{ for } a_n = \frac{1}{250}, \frac{1}{50}, \frac{1}{10}, \ldots 625$
15. $a_{12} \text{ for } a_n = 96, 48, 24, \ldots \frac{3}{4}$
16. $a_1 = 8, r = \frac{1}{2}$, $n = 9$

17. $a_1 = -3125, r = -\frac{1}{5}, n = 9$
18. $a_1 = 3, r = \frac{1}{10}, n = 8$

Write an equation for the $n$th term of each geometric sequence.

19. $1, 4, 16, \ldots a_n = (4)^{n-1}$
20. $-1, -5, -25, \ldots a_n = (-5)^{n-1}$

21. $1, \frac{1}{2}, \frac{1}{4}, \ldots a_n = \left(\frac{1}{2}\right)^{n-1}$
22. $-3, -6, -12, \ldots a_n = -3(2)^{n-1}$
23. $7, -14, 28, \ldots a_n = 7(-2)^{n-1}$

Find the geometric means in each sequence.

24. $3, \ldots 2, \ldots \frac{768}{12}, 48, 192$
25. $5, .2, \ldots .2, \ldots .1280 \pm 20, 80, \pm 320$

26. $144, \ldots .7, \ldots .9$
27. $37, 500, \ldots .7, \ldots .12, \pm 720, 36, \pm 18$

28. $\pm 750, 1500, \pm 300, 60$

29. BIOLOGY A culture initially contains 200 bacteria. If the number of bacteria doubles every 2 hours, how many bacteria will be in the culture at the end of 12 hours? 12,800

30. LIGHT If each foot of water in a lake screens out 60% of the light above, what percent of the light passes through 5 feet of water? 1.024%

31. INVESTING Raul invests $3000 in a savings account that earns 5% interest compounded annually. How much money will he have in the account at the end of 5 years? $1276.28

1. INVESTMENT Beth deposits $1500 into a retirement account that pays an APR of 8% compounded yearly. Assuming Beth makes no withdrawals, how much money will she have in her account after 20 years? $6991.44

4. MONGESE A population of mongoose has been growing by 20% every year. If the initial population was 5000 mongoose, what is the size of the mongoose population after n years? How many years will it take, roughly, for the mongoose population to exceed 10,000 mongoose? $5000(1.2)^n$, 4 years

2. CAKE Lauren has a piece of cake. She decides she wants to save some for later, so she eats half of it. Each time she returns to what remains, she only eats half of what is left. After her 8th serving of even smaller portions of cake, how much of the piece remains? $(0.5)^8$ of the original piece.

3. MOORE’S LAW Gordon Moore, co-founder of Intel, suggested that the number of transistors on a square inch of integrated circuit in a computer chip would double every 18 months. Assuming Moore’s law is true, how many times as many transistors would you expect on a square inch of integrated circuit every 18 months for the next 6 years? $2, 4, 8, 16$

5. Do the entries in the “Number of Cells” row form a geometric series? If so, find $r$.
   Yes; $r = 2$

6. Write an expression to find the $n$th term of the sequence. $a_n = 2^{n-1}$

7. Find the number of cells after 100 divisions. $6.34 \times 10^{39}$
### 11-3 Enrichment

**Half the Distance**

Suppose you are 200 feet from a fixed point, P. Suppose that you are able to move to the halfway point in one minute, to the next halfway point one minute after that, and so on.

![Diagram showing a person moving from point P to halfway points at 100 feet, 150 feet, and 175 feet]

An interesting sequence results because according to the problem, you never actually reach the point P, although you do get arbitrarily close to it.

You can compute how long it will take to get within some specified small distance of the point. On a calculator, you enter the distance to be covered and then count the number of successive divisions by 2 necessary to get within the desired distance.

**Example**

How many minutes are needed to get within 0.1 foot of a point 200 feet away?

Count the number of times you divide by 2.

Enter: 200 ÷ 2 ÷ 2 ÷ 2 ÷ 2, and so on

Result: 0.0976562

You divided by 2 eleven times. The time needed is 11 minutes.

**Exercises**

Use the method illustrated above to solve each problem.

1. If it is about 2500 miles from Los Angeles to New York, how many minutes would it take to get within 0.1 mile of New York? How far from New York are you at that time? 5 minutes, 0.0762934 mile

2. If it is 25,000 miles around Earth, how many minutes would it take to get within 0.5 mile of the full distance around Earth? How far short would you be? 16 minutes, 0.3814697 mile

3. If it is about 250,000 miles from Earth to the Moon, how many minutes would it take to get within 0.5 mile of the Moon? How far from the surface of the Moon would you be? 19 minutes, 0.4768372 mile

4. If it is about 30,000,000 feet from Honolulu to Miami, how many minutes would it take to get to within 1 foot of Miami? How far from Miami would you be at that time? 25 minutes, 0.8940967 mile

5. If it is about 8,000,000 miles to the sun, how many minutes would it take to get within 500 miles of the sun? How far from the sun would you be at that time? 18 minutes, 354.768846 miles

### 11-4 Lesson Reading Guide

**Geometric Series**

**Get Ready for the Lesson**

Read the introduction to Lesson 11-4 in your textbook.

- Suppose that you e-mail the joke on Monday to five friends, rather than three, and that each of those friends e-mails it to five friends on Tuesday, and so on. Write a sum that shows that total number of people, including yourself, who will have read the joke by Thursday. Write the sum using plus signs rather than sigma notation. Do not actually find the sum. 1 + 5 + 25 + 125
- Use exponents to rewrite the sum you found above. (Use an exponent in each term, and use the same base for all terms.) 5⁰ + 5¹ + 5² + 5³

**Read the Lesson**

1. Consider the formula \( S_n = \frac{a_1 (1 - r^n)}{1 - r} \).

   a. What is this formula used to find? the sum of the first \( n \) terms of a geometric series

   b. What do each of the following represent?

   \( S_n \): the sum of the first \( n \) terms

   \( a_1 \): the first term

   \( r \): the common ratio

   c. Suppose that you want to use the formula to evaluate \( 3 - 1 + \frac{1}{3} - \frac{1}{9} + \frac{1}{27} \). Indicate the values you would substitute into the formula in order to find \( S_n \). (Do not actually calculate the sum.)

   \[ n = 5 \quad a_1 = \frac{3}{2} \quad r = -\frac{1}{3} \quad r^n = \left( -\frac{1}{3} \right)^5 = -\frac{1}{243} \]

   d. Suppose that you want to use the formula to evaluate the sum \( \sum_{n=1}^{5} 2^n - 1 \). Indicate the values you would substitute into the formula in order to find \( S_n \). (Do not actually calculate the sum.)

   \[ n = 6 \quad a_1 = 8 \quad r = -2 \quad r^n = \left( -2 \right)^6 = 64 \]

**Remember What You Learned**

2. This lesson includes three formulas for the sum of the first \( n \) terms of a geometric series. All of these formulas have the same denominator and have the restriction \( r \neq 1 \). How can this restriction help you to remember the denominator in the formulas?

   Sample answer: If \( r = 1 \), then \( r - 1 = 0 \). Because division by 0 is undefined, a formula with \( r - 1 \) in the denominator will not apply when \( r = 1 \).
Geometric Series

A geometric series is the indicated sum of consecutive terms of a geometric sequence.

Example 1
Find the sum of the first four terms of the geometric sequence for which \(a_1 = 120\) and \(r = \frac{1}{3}\).

\[ S_4 = \frac{a_1(1 - r^4)}{1 - r} \]

Since the sum is a geometric series, you can use the sum formula.

\[ S_4 = \frac{120(1 - \left(\frac{1}{3}\right)^4)}{1 - \frac{1}{3}} \]

Use a calculator.

The sum of the series is 177.78.

The sum of the series is 1457.33.

Example 2
Find the sum of the geometric series \(\sum_{k=1}^{\infty} 4 \cdot 3^k - 1\).

Since the sum is a geometric series, you can use the sum formula.

\[ S_n = \frac{a_1(1 - r^n)}{1 - r} \]

The sum of the series is 7.

Example 3
Find \(a_1\) in a geometric series for which \(S_n = 796.875\), \(r = \frac{1}{2}\), and \(n = 8\).

First use the sum formula to find \(S_n\).

\[ S_n = \frac{a_1(1 - r^n)}{1 - r} \]

Since \(a_4 = a_1 \cdot r^3\), \(a_4 = 400\left(\frac{1}{2}\right)^3 = 50\). The fourth term of the series is 50.

Exercises

Find \(S_n\) for each geometric series described.

1. \(a_1 = 2, a_n = 486, r = 3\)  
   \[ S_n = \frac{a_1(1 - r^n)}{1 - r} \]
   \[ 728 \]
   \[ 2325 \]
   \[ 156.24 \]

2. \(a_1 = 1200, a_n = 75, r = \frac{1}{2}\)  
   \[ S_n = \frac{a_1(1 - r^n)}{1 - r} \]
   \[ 244 \]
   \[ 518 \]
   \[ 2730 \]

3. \(a_1 = 2, a_n = 6, n = 4\)  
   \[ S_n = \frac{a_1(1 - r^n)}{1 - r} \]
   \[ 68.75 \]
   \[ 1275 \]
   \[ 87381.25 \]

Find the sum of each geometric series.

10. \(6 + 18 + 54 + \ldots\) to 6 terms
   \[ S_n = \frac{6(1 - 3^6)}{1 - 3} \]
   \[ 2184 \]
   \[ 255.75 \]

11. \(\frac{\frac{1}{2}}{\frac{1}{2}} + 1 + \ldots\) to 10 terms
   \[ S_n = \frac{\left(\frac{1}{2}\right)^{10} - \frac{1}{2}}{\frac{1}{2} - 1} \]
   \[ 496 \]
   \[ 381 \]
11-4 Skills Practice

Geometric Series

Find \( S_n \) for each geometric series described.

1. \( a_1 = 2, a_k = 162, r = 3 \) \( 242 \)
2. \( a_1 = 4, a_k = 12,500, r = 5 \) \( 15,624 \)
3. \( a_1 = 1, a_k = -1, r = -1 \) \( 0 \)
4. \( a_1 = 4, a_k = 256, r = 2 \) \( 172 \)
5. \( a_1 = 1, a_k = 729, r = -3 \) \( 547 \)
6. \( a_1 = 2, r = -4, n = 5 \) \( 410 \)
7. \( a_1 = -8, r = 2, n = 4 \) \( -120 \)
8. \( a_1 = 3, r = -2, n = 12 \) \( -4095 \)
9. \( a_1 = 8, r = 3, n = 5 \) \( 968 \)
10. \( a_1 = 6, a_k = \frac{3}{4}, r = \frac{1}{2} \) \( \frac{93}{8} \)
11. \( a_1 = 8, r = \frac{1}{2}, n = 7 \) \( \frac{127}{8} \)
12. \( a_1 = 2, r = -\frac{1}{2}, n = 6 \) \( \frac{21}{16} \)

Find the sum of each geometric series.

13. \( 4 + 8 + 16 + \ldots \) to 5 terms \( 124 \)
14. \( -1 - 3 - 9 - \ldots \) to 6 terms \( -364 \)
15. \( 3 + 6 + 12 + \ldots \) to 5 terms \( 93 \)
16. \( -15 + 30 + 60 + \ldots \) to 7 terms \( -645 \)
17. \( \sum_{k=1}^{n} (-3)^n - 1 \) \( 40 \)
18. \( \sum_{k=1}^{n} (-2)^n - 1 \) \( 11 \)
19. \( \sum_{k=1}^{n} \left( \frac{1}{2} \right)^n - 1 \) \( 40 \)
20. \( \sum_{k=1}^{n} 2(-3)^n - 1 \) \( 327 \)

Find the indicated term for each geometric series described.

21. \( S_n = 1275, a_k = 640, r = 2; a_1 \) \( 5 \)
22. \( S_n = -40, a_k = -54, r = -3; a_1 \) \( 2 \)
23. \( S_n = 99, n = 5, r = -\frac{1}{2}; a_1 \) \( 144 \)
24. \( S_n = 35,360, n = 8, r = 3; a_1 \) \( 12 \)

Chapter 11

30

Glencoe Algebra 2
1. BASE 10  When the common ratio of a geometric series is 10, the sum is sometimes easier to compute because we use a decimal number system. For example, what is the sum of \(1 + 10 + 10^2 + 10^3 + 10^4 + 10^5\)?

\[\frac{10^6 - 1}{10 - 1} = 111,111\]

2. INVITATIONS  Amanda wants to host a party. She invites 3 friends and tells each of them to invite 3 of their friends. The 3 friends do invite 3 others and ask each of them to invite 3 more people. This invitation process goes on for 5 generations of invitations. Including herself, how many people can Amanda expect at her party?

\[3^n - 1 \div 7\] The answer is 364.

3. TRAINING  Arnold lifts weights. He does three bench press workouts each week. For each, he lifts a weight 12 times. The first week he starts with 50 pounds. Each week he increases the weight that he lifts by 10. After 10 weeks, what is the total amount of weight that Arnold has lifted during his bench press workouts? Round your answer to the nearest pound.

\[28,687\text{ lb}\]

4. TEACHING  A teacher teaches 8 students how to fold an origami model. Each of those students goes on to teach 8 students of their own how to fold the same model. If this teaching process goes on for \(n\) generations, how many people will know how to fold the origami model?

<table>
<thead>
<tr>
<th>Generation</th>
<th>Number of People Taught</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>8</td>
</tr>
<tr>
<td>2</td>
<td>64</td>
</tr>
<tr>
<td>3</td>
<td>512</td>
</tr>
<tr>
<td>4</td>
<td>4,096</td>
</tr>
<tr>
<td>5</td>
<td>32,768</td>
</tr>
</tbody>
</table>

5. What is Mary's salary for her 6th year?

\[50000(1.07)^{6-1}\]

6. Use sigma notation to give an expression for the total income she will receive from the university after \(n\) years.

\[50000 \cdot 1.07^{n-1}\]

7. What will be her total income from the university after 20 years?

\[\$2,049,774.62\]

8. Annuities

An annuity is a fixed amount of money payable at given intervals. For example, suppose you wanted to set up a trust fund so that $30,000 could be withdrawn each year for 14 years before the money ran out. Assume the money can be invested at 9%.

You must find the amount of money that needs to be invested. Call this amount \(A\). After the third payment, the amount left is

\[1.09(1.09A - 30,000(1 + 1.09)) - 30,000 = 1.09A - 30,000(1 + 1.09) + 1.09^2\]

The results are summarized in the table below.

<table>
<thead>
<tr>
<th>Payment Number</th>
<th>Number of Dollars Left After Payment</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>(A - 30,000)</td>
</tr>
<tr>
<td>2</td>
<td>(1.09A - 30,000(1 + 1.09))</td>
</tr>
<tr>
<td>3</td>
<td>(1.09^2A - 30,000(1 + 1.09 + 1.09^2))</td>
</tr>
</tbody>
</table>

1. Use the pattern shown in the table to find the number of dollars left after the fourth payment.

\[1.09^3A - 30,000(1 + 1.09 + 1.09^2 + 1.09^3)\]

2. Find the amount left after the tenth payment.

\[1.09^9A - 30,000(1 + 1.09 + 1.09^2 + 1.09^3 + \ldots + 1.09^9)\]

The amount left after the 14th payment is \(1.09^{13}A - 30,000(1 + 1.09 + 1.09^2 + \ldots + 1.09^{13})\). However, there should be no money left after the 14th and final payment.

\[1.09^{13}A - 30,000(1 + 1.09 + 1.09^2 + \ldots + 1.09^{13}) = 0\]

Notice that \(1 + 1.09 + 1.09^2 + \ldots + 1.09^{13}\) is a geometric series where \(a_1 = 1, a_2 = 1.09, n = 14\) and \(r = 1.09\).

Using the formula for \(S_n\),

\[1 + 1.09 + 1.09^2 + \ldots + 1.09^{13} = \frac{1 - 1.09^{14}}{1 - 1.09} = \frac{1.09^{14} - 1}{0.09} = \frac{1.09^{14}}{0.09} = 13,382.29\]

3. Show that when you solve for \(A\) you get

\[A = \frac{30,000(1.09^{14} - 1)}{0.09} = 133,822.89\]

Therefore, to provide $30,000 for 14 years where the annual interest rate is 9%, you need $133,822.89 dollars.

4. Use a calculator to find the value of \(A\) in problem 3.$254,607.

In general, if you wish to provide \(P\) dollars for each of \(n\) years at an annual rate of \(r\%,\) you need \(A\) dollars where

\[\left(1 + \frac{r}{100}\right)^n - A = P\left(1 + \frac{r}{100}\right)^{n-1} + \left(1 + \frac{r}{100}\right)^{n-2} + \ldots + \left(1 + \frac{r}{100}\right)^1 = 0\]

You can solve this equation for \(A\) given \(P, n,\) and \(r\).
11-5 Lesson Reading Guide

Infinite Geometric Series

Get Ready for the Lesson

Read the introduction to Lesson 11-5 in your textbook.

Note the following powers of 0.6: 0.6^1 = 0.6; 0.6^2 = 0.36; 0.6^3 = 0.216; 0.6^4 = 0.1296; 0.6^5 = 0.07776; 0.6^6 = 0.046656; 0.6^7 = 0.0279936. If a ball is dropped from a height of 10 feet and bounces back to 60% of its previous height on each bounce, after how many bounces will it bounce back to a height of less than 1 foot? 5 bounces

Read the Lesson

1. Consider the formula S = \frac{a_1}{1 - r}
   a. What is the formula used to find the sum of an infinite geometric series?
   b. What do each of the following represent?
      - S: the sum
      - a_1: the first term
      - r: the common ratio
   c. For what values of r does an infinite geometric sequence have a sum? −1 < r < 1
   d. Rewrite your answer for part c as an absolute value inequality. |r| < 1

2. For each of the following geometric series, give the values of a_1 and r. Then state whether the sum of the series exists. (Do not actually find the sum.)
   a. \frac{2}{3} + \frac{2}{9} + \frac{2}{27} + \ldots
      \quad a_1 = \frac{2}{3} \quad r = \frac{1}{3}
      \quad Does the sum exist? yes
   b. 2 - 1 + \frac{1}{2} - \frac{1}{4} + \ldots
      \quad a_1 = 2 \quad r = \frac{1}{2}
      \quad Does the sum exist? yes
   c. \sum_{i=1}^{\infty} 2^i
      \quad a_1 = 2 \quad r = 2
      \quad Does the sum exist? no

Remember What You Learned

3. One good way to remember something is to relate it to something you already know. How can you use the formula S_n = \frac{a_1(1 - r^n)}{1 - r} that you learned in Lesson 11-4 for finding the sum of a geometric series to help you remember the formula for finding the sum of an infinite geometric series? Sample answer: If −1 < r < 1, then as n gets large, r^n approaches 0, so 1 − r^n approaches 1. Therefore, S_n approaches \frac{a_1}{1 - r}, or \frac{a_1}{1 - r}.

Example

Find the sum of each infinite geometric series, if it exists.

a. 75 + 15 + 3 + \ldots
   First, find the value of r to determine if the sum exists. a_1 = 75 and a_2 = 15, so
   \quad r = \frac{15}{75} or \frac{1}{5}. Since |\frac{1}{5}| < 1, the sum exists. Now use the formula for the sum of an infinite geometric series.
   \quad S = \frac{a_1}{1 - r}
   \quad S = \frac{75}{1 - \frac{1}{5}}
   \quad S = \frac{75}{\frac{4}{5}}
   \quad S = \frac{75}{1} or 75
   \quad a_1 = 75, r = \frac{1}{5}
   The sum of the series is 75.

b. \sum_{n=1}^{\infty} \frac{48}{n+1} \cdot \left(\frac{1}{3}\right)^n
   In this infinite geometric series, a_1 = \frac{48}{2} and r = \frac{1}{3}.
   \quad S = \frac{\frac{48}{2}}{1 - \frac{1}{3}}
   \quad S = \frac{48}{1 - \frac{1}{3}}
   \quad S = \frac{48}{\frac{2}{3}}
   \quad S = 36
   Thus, \sum_{n=1}^{\infty} \frac{48}{n+1} \cdot \left(\frac{1}{3}\right)^n = 36.

Exercises

Find the sum of each infinite geometric series, if it exists.

1. a_1 = 7, r = \frac{5}{3}
2. 1 + \frac{5}{4} + \frac{25}{16} + \ldots
3. a_1 = 4, r = \frac{1}{2}
   -18 \frac{2}{3} does not exist

4. \frac{2}{3} + \frac{2}{9} + \frac{2}{27} + \ldots
5. 15 + 10 + 6\frac{2}{3} + \ldots
6. 18 - 9 + 4\frac{1}{2} - 2\frac{1}{4} + \ldots
   1 \frac{1}{3}
   45
   12

7. \frac{1}{10} + \frac{1}{20} + \frac{1}{40} + \ldots
8. 1000 + 800 + 640 + \ldots
9. 6 - 12 + 24 - 48 + \ldots
   1 \frac{5}{9}
   5000
   does not exist

10. \sum_{n=1}^{\infty} 50 \left(\frac{1}{2}\right)^n
11. \sum_{n=1}^{\infty} 22 \left(\frac{1}{2}\right)^{n-1}
12. \sum_{n=1}^{\infty} 24 \left(\frac{7}{12}\right)^{n-1}
   250
   14 \frac{2}{3}
   57 \frac{3}{5}
11-5

Study Guide and Intervention (continued)

Infinite Geometric Series

Repeating Decimals A repeating decimal represents a fraction. To find the fraction, write the decimal as an infinite geometric series and use the formula for the sum.

Example

Write each repeating decimal as a fraction.

\[ a. \ 0.\bar{42} \]

Write the repeating decimal as a sum.

\[ 0.\bar{42} = 0.42424242... = \frac{42}{100} + \frac{42}{10000} + ... \]

In this series \( a_1 = \frac{42}{100} \) and \( r = \frac{1}{100} \).

\[ S = \frac{\frac{42}{100}}{1 - \frac{1}{100}} = \frac{42}{100 - 1} = \frac{42}{99} \]

Thus, \( 0.\bar{42} = \frac{14}{33} \).

\[ b. \ 0.5\bar{24} \]

Let \( S = 0.5\bar{24} \).

\[ S = 0.5\bar{24} = 0.52424242... \]

Multiply each side by 1000.

\[ 1000S = 524.24242424... \]

Multiply each side by 10.

\[ 999S = 519 \]

Subtract the third equation from the second equation, simplify.

\[ S = \frac{519}{999} \]

Thus, \( 0.5\bar{24} = \frac{173}{330} \).

Exercises

Write each repeating decimal as a fraction.

1. \( 0.\bar{3} = \frac{1}{3} \)
2. \( 0.\bar{2} = \frac{2}{9} \)
3. \( 0.\bar{3}0 \)
4. \( 0.\bar{37} = \frac{29}{33} \)
5. \( 0.\bar{70} = \frac{7}{9} \)
6. \( 0.\bar{34} = \frac{6}{11} \)
7. \( 0.\bar{75} = \frac{25}{33} \)
8. \( 0.\bar{18} = \frac{2}{11} \)
9. \( 0.\bar{65} = \frac{65}{99} \)
10. \( 0.\bar{77} = \frac{77}{99} \)
11. \( 0.\bar{072} = \frac{4}{55} \)
12. \( 0.\bar{045} = \frac{1}{22} \)
13. \( 0.\bar{06} = \frac{1}{15} \)
14. \( 0.\bar{0138} = \frac{23}{1665} \)
15. \( 0.\bar{0138} = \frac{46}{3333} \)
16. \( 0.\bar{081} = \frac{9}{110} \)
17. \( 0.\bar{077} = \frac{27}{110} \)
18. \( 0.\bar{83} = \frac{24}{55} \)
19. \( 0.\bar{57} = \frac{49}{90} \)
20. \( 0.\bar{863} = \frac{19}{22} \)

11-5

Skills Practice

Find the sum of each infinite geometric series, if it exists.

1. \( a_1 = 1, r = \frac{1}{2} \)
2. \( a_1 = 5, r = \frac{2}{5} \)
3. \( a_1 = 8, r = 2 \) does not exist
4. \( a_1 = 6, r = \frac{1}{2} \)
5. \( 4 \cdot 2 + 1 + \frac{1}{2} + ... \) \( 8 \)
6. \( 6 \cdot 40 \cdot 180 + 80 - 20 + ... \) \( 405 \)
7. \( 5 + 10 + 20 + ... \) does not exist
8. \( -336 + 84 - 21 + ... \) \( -268.8 \)
9. \( 125 + 25 + 5 + ... \) \( 156.25 \)
10. \( 10 \cdot 9 - 1 + \frac{1}{9} - ... \) \( 81 \)
11. \( \frac{3}{4} + \frac{2}{4} + \frac{27}{4} + ... \) does not exist
12. \( \frac{1}{4} + \frac{1}{5} + \frac{1}{27} + ... \) \( \frac{1}{2} \)
13. \( 5 + 2 + 0.8 + ... \) \( \frac{25}{3} \)
14. \( 14 \cdot 9 + 6 + 4 + ... \) \( 27 \)
15. \( \sum_{n=1}^{\infty} \frac{10}{2}^{n-1} \) \( 20 \)
16. \( \sum_{n=1}^{\infty} \frac{6}{7}^{n-1} \) \( \frac{9}{2} \)
17. \( \sum_{n=1}^{\infty} \frac{2}{5}^{n-1} \) \( 25 \)
18. \( \sum_{n=1}^{\infty} \left( \frac{4}{3} \right)^{n-1} \) \( -2 \)

Write each repeating decimal as a fraction.

19. \( 0.\bar{4} = \frac{4}{9} \)
20. \( 0.\bar{3} = \frac{3}{9} \)
21. \( 0.\bar{27} = \frac{27}{11} \)
22. \( 0.\bar{67} = \frac{67}{99} \)
23. \( 0.\bar{04} = \frac{4}{11} \)
24. \( 0.\bar{375} = \frac{125}{333} \)
25. \( 0.\bar{011} = \frac{11}{111} \)
11-5 Practice

 Infinite Geometric Series

Find the sum of each infinite geometric series, if it exists.

1. \( a_1 = 35, r = \frac{2}{5} \) \( \frac{49}{5} \)
2. \( a_1 = 26, r = \frac{1}{3} \) \( \frac{52}{3} \)
3. \( a_1 = 98, r = -\frac{2}{4} \) \(-\frac{56}{2} \)
4. \( a_1 = 42, r = \frac{6}{5} \) does not exist
5. \( a_1 = 114, r = -\frac{3}{2} \) \(-\frac{70}{2} \)
6. \( a_1 = 500, r = \frac{1}{5} \) \( \frac{625}{5} \)
7. \( a_1 = 135, r = -\frac{1}{2} \) \(-\frac{90}{2} \)
8. \( a_1 = 18, r = -\frac{6}{2} + \cdots \) \( \frac{27}{2} \)
9. \( 2 + 6 + 12 + \cdots \) does not exist
10. \( 6 + 4 + \frac{4}{3} + \cdots \) \( \frac{18}{3} \)
11. \( \frac{1}{15} \) 
12. \( \frac{1}{2} + \frac{1}{3} + \cdots \) \( \frac{100}{9} \)
13. \( 100 + 90 + 80 + \cdots \) \( \frac{125}{3} \)
14. \( 60.5 + 26 + \frac{15}{5} + \cdots \) \( 1 \)
15. \( 18.0 + 0.08 + 0.008 + \cdots \) \( \frac{9}{8} \)
16. \( \frac{1}{12} = \frac{1}{6} + \frac{1}{3} + \cdots \) \( \frac{50}{3} \)
17. \( \frac{3}{9} + \frac{2}{12} + \cdots \) \( \frac{21}{4} \)
18. \( \frac{2}{4} + \frac{2}{6} + \cdots \) \( \frac{1}{6} \)
19. \( \frac{3}{2} + \frac{2}{3} + \cdots \) \( \frac{4}{3} \)
20. \( \frac{3}{2} + \frac{2}{4} + \cdots \) \( \frac{4}{3} \)
21. \( \frac{2}{1} \)
22. \( \frac{1}{1} \)
23. \( \frac{1}{1} \)
24. \( \frac{1}{1} \)
25. \( \frac{1}{1} \)

Write each repeating decimal as a fraction.

27. \( 0.\overline{6} \) \( \frac{2}{3} \)
28. \( 0.\overline{5} \) \( \frac{5}{11} \)
29. \( 0.\overline{4} \) \( \frac{4}{9} \)
30. \( 0.27 \) \( \frac{3}{11} \)
31. \( 0.245 \) \( \frac{9}{37} \)
32. \( 0.32 \) \( \frac{29}{93} \)
33. \( 0.989 \) \( \frac{110}{111} \)
34. \( 0.15 \) \( \frac{50}{333} \)

35. PENDULUMS On its first swing, a pendulum travels 8 feet. On each successive swing, the pendulum travels \( \frac{1}{2} \) the distance of its previous swing. What is the total distance traveled by the pendulum when it stops swinging? 60 ft

36. ELASTICITY A ball dropped from a height of 10 feet bounces back 2/3 of that distance. With each successive bounce, the ball continues to reach \( \frac{2}{3} \) of its previous height. What is the total vertical distance (both up and down) traveled by the ball when it stops bouncing? 190 ft (Hint: Add the total distance the ball falls to the total distance it rises.)

11-5 Word Problem Practice

 Infinite Geometric Series

1. PARADOX If the formula for the sum of the infinite geometric series is applied to the series whose first term is 1 and common ratio is 2, the result is the equation \( -1 - 1 + 2 + 4 + 8 + \cdots \). Is this equality really true? Explain.
   No, it is not true. An infinite geometric series must have \( r < 1 \) to have a sum.

2. BOUNDS Can the sum of an infinite geometric series whose first term is 1 be as large as we wish?
   Yes, it can. The sum is \( 1 - r \), where \( r \) is the common ratio. This expression increases without bound as \( r \) approaches 1 from below.

3. BASES The infinite repeating decimal 0.999... is equal to 1. This can be shown by using the sum of a geometric series with common ratio \( \frac{1}{10} \) and first term \( \frac{9}{10} \).
   In a similar vein, compute the sum of the infinite geometric series \( \frac{b - 1}{b} \) + \( \frac{b - 1}{b^2} \) + \( \frac{b - 1}{b^3} \) + ... , where \( b \) is a positive integer greater than 1. How is this sum related to the fact that 0.999... = 1?
   It is similar to 0.999... in base \( b \).

4. CLIMBING A robot is designed to climb a wall each time a button is pressed. The first time the button is pressed, it climbs 10 feet. Each time after that, the robot climbs only 75% of what it climbed the last time. What is the smallest upper limit on how high the robot can climb?
   40 ft

INSTALLMENTS For Exercises 5-7, use the following information.

Jade lends Jack a 100-pound chunk of pure gold for one year. After one year, she wants to start getting the gold back. One year later, Jack begins returning the gold, by giving Jade 1 pound of gold. The next day, Jack gives her 0.99 pounds of gold. The next day, Jack gives her 0.999 pounds of gold. Each successive day, Jack gives 0.99 times as much gold as the previous day.

5. How much gold does Jade get back on the nth day that Jack begins returning the gold?
   (0.99)^n-1 lb

6. How much gold has Jade received after 10 days? 100 days? Infinitely many days? Round your answers to the nearest hundredth of a pound.
   9.56 lb after 10 days; 63.40 lb after 100 days; 100 lb after infinitely many days

7. Will Jade have all her gold back at any specific date in the future? Explain.
   No. At the rate Jack is returning the gold, there will always be a small amount of the gold that will never be returned.
11-5 Enrichment

Infinite Continued Fractions

Some infinite expressions are actually equal to real numbers. The infinite continued fraction at the right is one example:

\[ x = 1 + \frac{1}{x} \]

If you use \( x \) to stand for the infinite fraction, then the entire denominator of the first fraction on the right is also equal to \( x \). This observation leads to the following equation:

\[ x = 1 + \frac{1}{x} \]

Write a decimal for each continued fraction.

1. \( 1 + \frac{1}{2} \) 2 2. \( 1 + \frac{1}{\frac{1}{2}} \) 1.5 3. \( 1 + \frac{1}{\frac{1}{1.5}} \) 1.665

4. \( 1 + \frac{1}{\frac{1}{1.6}} \) 1.6 5. \( 1 + \frac{1}{\frac{1}{1.625}} \) 1.625

6. The more terms you add to the fractions above, the closer their value approaches the value of the infinite continued fraction. What value do the fractions seem to be approaching? **about 1.6**

7. Rewrite \( x = 1 + \frac{1}{x} \) as a quadratic equation and solve for \( x \).

\[ x^2 - x - 1 = 0; \quad x = \frac{1 \pm \sqrt{5}}{2}; \quad x \approx 1.618 \text{ or } -0.618 \]

(The positive root is the value of the infinite fraction, because the original fraction is not clearly negative.)

8. Find the value of the following infinite continued fraction.

\[ 3 + \frac{1}{3 + \frac{1}{3 + \frac{1}{3 + \frac{1}{3 + \frac{1}{3 + \cdots}}}}} \]

\[ x = 3 + \frac{1}{x}; \quad x = 3 + \frac{1}{3 + \frac{1}{3 + 1/3 + \cdots}} \]

\[ x = 4 \] or **about 3.30**

11-6 Lesson Reading Guide

Recursion and Special Sequences

Get Ready for the Lesson

Read the introduction to Lesson 11-6 in your textbook.

What are the next three numbers in the sequence that gives the number of sheets corresponding to each month? **8, 13, 21**

Read the Lesson

1. Consider the sequence in which \( a_1 = 4 \) and \( a_n = 2a_{n-1} + 1 \).

   a. Explain why this is a recursive formula. **Sample answer:** Each term is found from the value of the previous term.

   b. Explain in your own words how to find the first four terms of this sequence. (Do not actually find any terms after the first.) **Sample answer:** The first term is 4. To find the second term, double the first term and add 5. To find the third term, double the second term and add 5. To find the fourth term, double the third term and add 5.

   c. What happens to the terms of this sequence as \( n \) increases? **Sample answer:** They keep getting larger and larger.

2. Consider the function \( f(x) = 3x - 1 \) with an initial value of \( x_0 = 2 \).

   a. What does it mean to iterate this function?

   **to compose the function with itself repeatedly**

   b. Fill in the blanks to find the first three iterates. The blanks that follow the letter \( x \) are for subscripts.

   \[ x_1 = f(x_0) = f(2) = 3(2) - 1 = 5 \]

   \[ x_2 = f(x_1) = f(5) = 3(5) - 1 = 14 \]

   \[ x_3 = f(x_2) = f(14) = 3(14) - 1 = 40 \]

   c. As this process continues, what happens to the values of the iterates? **Sample answer:** They keep getting larger and larger.

Remember What You Learned

3. Use a dictionary to find the meanings of the words **recurrent** and **iterate**. How can the meanings of these words help you to remember the meaning of the mathematical terms **recursion** and **iteration**? How are these ideas related? **Sample answer:** Recurrent means happening repeatedly, while iterate means to repeat a process or operation. A recursive formula is used repeatedly to find the value of one term of a sequence based on the previous term. Iteration means to compose a function with itself repeatedly. Both ideas have to do with repetition—doing the same thing over and over again.
11-6 Study Guide and Intervention

Recursion and Special Sequences

Special Sequences In a recursive formula, each succeeding term is formulated from one or more previous terms. A recursive formula for a sequence has two parts:
1. the value(s) of the first term(s), and
2. an equation that shows how to find each term from the term(s) before it.

Example: Find the first five terms of the sequence in which $a_1 = 6$, $a_2 = 10$, and $a_n = 2a_{n-2}$ for $n \geq 3$.

- $a_1 = 6$
- $a_2 = 10$
- $a_3 = 2a_2 = 2(10) = 20$
- $a_4 = 2a_3 = 2(20) = 40$
- $a_5 = 2a_4 = 2(40) = 80$

The first five terms of the sequence are 6, 10, 20, 40, 80.

Exercises

Find the first five terms of each sequence.

1. $a_1 = 1$, $a_2 = 1$, $a_n = 2(a_{n-1} + a_{n-2})$, $n \geq 3$
2. $a_1 = 1$, $a_2 = 1$, $a_n = \frac{1}{1 + a_{n-1}}$, $n \geq 2$
3. $a_1 = 1$, $a_2 = 5$, $a_n = a_{n-1} + 2(n-2)$, $n \geq 3$
4. $a_1 = 1$, $a_2 = 3$, $a_n = a_{n-1} + a_{n-2}$, $n \geq 3$
5. $a_1 = 1$, $a_2 = 1$, $a_n = (n-1)a_{n-1}$, $n \geq 2$
6. $a_1 = 7$, $a_2 = 4a_1 - 1$, $n \geq 2$
7. $a_1 = 3$, $a_2 = 4$, $a_n = 2a_{n-2} + 3a_{n-1}$, $n \geq 3$
8. $a_1 = 0.5$, $a_2 = a_{n-1} + 2a_{n-2}$, $n \geq 2$
9. $a_1 = 2$, $a_2 = 10$, $a_n = \frac{a_{n-2}}{a_{n-1}}$, $n \geq 3$
10. $a_1 = 100$, $a_2 = \frac{a_{n-1} - 1}{n}$, $n \geq 2$

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11-6 Study Guide and Intervention (continued)

Iteration Combining composition of functions with the concept of recursion leads to the process of iteration. Iteration is the process of composing a function with itself repeatedly.

Example: Find the first three iterates of $f(x) = 4x - 5$ for an initial value of $x_0 = 2$.

- First iterate: $x_1 = f(x_0) = 4(2) - 5 = 3$
- Second iterate: $x_2 = f(x_1) = 4(3) - 5 = 7$
- Third iterate: $x_3 = f(x_2) = 4(7) - 5 = 27$

Exercises

Find the first three iterates of each function for the given initial value.

1. $f(x) = \frac{x}{x - 1}$, $x_0 = 2$
2. $f(x) = x^2 - 3x$, $x_0 = 1$
3. $f(x) = \sqrt{x} + 1$, $x_0 = 4$
4. $f(x) = 3x - 2$, $x_0 = 1$
5. $f(x) = 2x - 4$, $x_0 = 2$
6. $f(x) = x + 2$, $x_0 = 3$
7. $f(x) = x^2 - 4x$, $x_0 = 1$
8. $f(x) = x^2 - 3x + 2$, $x_0 = 1$
9. $f(x) = x^2 - 2x + 1$, $x_0 = 1$
10. $f(x) = x^2 - 2x + 1$, $x_0 = 2$

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### 11-6 Skills Practice

**Recursion and Special Sequences**

Find the first five terms of each sequence.

1. \(a_1 = 4\), \(a_{n+1} = a_n + 7\)
   \(4, 11, 18, 25, 32\)

2. \(a_1 = 3\), \(a_{n+1} = a_n + 3\)
   \(-2, 1, 4, 7, 10\)

3. \(a_1 = 5\), \(a_{n+1} = 2a_n\)
   \(5, 10, 20, 40, 80\)

4. \(a_1 = -4\), \(a_{n+1} = -6 - a_n\)
   \(-4, 10, -4, 10, -4\)

5. \(a_1 = 1\), \(a_{n+1} = a_n + n\)
   \(1, 2, 4, 7, 11\)

6. \(a_1 = -1\), \(a_{n+1} = n - a_n\)
   \(-1, 2, 0, 3, 1\)

7. \(a_1 = -6\), \(a_{n+1} = a_n + n + 1\)
   \(-6, -4, -1, 3, 8\)

8. \(a_1 = 8\), \(a_{n+1} = a_n - n - 2\)
   \(8, 5, 1, -4, -10\)

9. \(a_1 = -3\), \(a_{n+1} = 2a_n + 7\)
   \(-3, 1, 9, 25, 57\)

10. \(a_1 = 4\), \(a_{n+1} = -2a_n - 5\)
    \(4, -13, 21, -47, 89\)

11. \(a_1 = 0\), \(a_{n+1} = a_n + 1 - a_{n-1}\)
    \(0, 1, 1, 2, 3\)

12. \(a_1 = -1\), \(a_{n+1} = a_n - a_{n-1}\)
    \(-1, -1, 0, 1, 1\)

13. \(a_1 = 3\), \(a_2 = 3, a_{n+1} = 2a_n + a_{n-1}\)
    \(3, -5, 23, -97, 411\)

**Find the first three iterates of each function for the given initial value.**

15. \(f(x) = 2x - 1, x_0 = 3\)
    \(5, 9, 17\)

16. \(f(x) = 5x - 3, x_0 = 2\)
    \(7, 32, 157\)

17. \(f(x) = 3x + 4, x_0 = -1\)
    \(1, 7, 25\)

18. \(f(x) = 4x + 7, x_0 = -5\)
    \(-13, -45, -173\)

19. \(f(x) = -x - 3, x_0 = 10\)
    \(-13, 10, -13\)

20. \(f(x) = -3x + 6, x_0 = 2\)
    \(-12, 42, -120\)

21. \(f(x) = -3x + 4, x_0 = 2\)
    \(-10, 46, 90\)

22. \(f(x) = 6x - 5, x_0 = 1\)
    \(1, 1, 1\)

23. \(f(x) = 3x + 1, x_0 = -4\)
    \(-27, -188, -1315\)

24. \(f(x) = x^2 - 3x, x_0 = 5\)
    \(10, 70, 4690\)

### 11-6 Practice

**Recursion and Special Sequences**

Find the first five terms of each sequence.

1. \(a_1 = 3\), \(a_{n+1} = a_n + 5\)
   \(3, 8, 13, 18, 23\)

2. \(a_1 = -2\), \(a_{n+1} = a_n + 8\)
   \(-7, 1, 9, 17, 25\)

3. \(a_1 = -3\), \(a_{n+1} = 2a_n + 2\)
   \(-3, -7, -19, -55, -163\)

4. \(a_1 = -8\), \(a_{n+1} = 10 - a_n\)
   \(-8, 18, -8, 18, -8\)

5. \(a_1 = 4\), \(a_{n+1} = n - a_n\)
   \(-3, -9, -27, -81, -243\)

6. \(a_1 = 3\), \(a_{n+1} = 3a_n\)
   \(-3, 9, -27, 81, -243\)

7. \(a_1 = 4\), \(a_{n+1} = 2a_n + 4\)
   \(-4, 8, 28, -80, 244\)

8. \(a_1 = 2\), \(a_{n+1} = -4a_n - 5\)
   \(-2, 13, 47, 193, 767\)

9. \(a_1 = 3\), \(a_{n+1} = 1 - a_n - a_{n-1}\)
   \(-2, -3, -3, -1\)

10. \(a_1 = 1\), \(a_{n+1} = 5a_n - 8a_{n-1}\)
    \(-1, 5, 29, -65\)

11. \(a_1 = -2\), \(a_{n+1} = 1 - a_n - a_{n-1}\)
    \(-2, -14, 34, -152\)

**Find the first three iterates of each function for the given initial value.**

13. \(f(x) = 3x + 4, x_0 = -1\)
   \(1, 7, 25\)

14. \(f(x) = 10x + 2, x_0 = -1\)
   \(-8, -78, -778\)

15. \(f(x) = 8 - x, x_0 = 1\)
   \(11, 41, 131\)

16. \(f(x) = 8 - x, x_0 = -3\)
   \(-1, -3, 11\)

17. \(f(x) = 4x + 5, x_0 = -1\)
   \(1, 9, 41\)

18. \(f(x) = 5x + 3, x_0 = -2\)
   \(5, 20, 105\)

19. \(f(x) = -8x + 9, x_0 = 1\)
   \(1, 1, 1\)

20. \(f(x) = -4x^3, x_0 = -1\)
    \(-4, -64, -16384\)

21. \(f(x) = x^2 + 3, x_0 = 3\)
    \(8, 63, 3968\)

22. \(f(x) = 2x^2, x_0 = 5\)
    \(50, 5000, 5000000\)

23. **INFLATION** Iterating the function \(f(x) = 1.02x\) gives the future cost of an item at a constant 2% inflation rate. Find the cost of a $3000 ring in five years at 5% inflation.

\[ \text{Cost in 5 years} = 3000 \times 1.02^5 = \$3592.75 \]

24. **FRACTIONS** For Exercises 24–27, use the following information.

Replacing each side of the square shown with the combination of segments below gives the figure to its right.

25. What is the perimeter of the original square? \(12\) in.

26. What is the perimeter of the new shape? \(20\) in.

27. If you repeat the process by replacing each side of the new shape by a proportional combination of 5 segments, what will be the perimeter of the third shape? \( \frac{332}{3}\) in.
11-6 Word Problem Practice

Recursion and Special Sequences

1. GEOMETRIC SEQUENCES The geometric sequence with first term $a$ and common ratio $r$ goes like this: $a, ar, ar^2, ar^3, \ldots$. It happens that this sequence can also be seen from the point of view of iterative sequences. What function $f(x)$ can be used to define the geometric sequence above iteratively? $f(x) = rx$

2. BACTERIA All the bacteria in a bacterial culture divide in two every hour. Also, every hour, 1,000 bacteria are removed from the culture. If the initial population consisted of 1,100 bacteria, what is the population size every hour for the next four hours? Starting with 1,100, the population increases to 1200, 1400, 1600, then 2600.

3. WORK The company that Robert works for has a policy where the number of hours you have to work one week depends on the number of hours worked the previous week. If you worked $h$ hours one week, then the next week you must work at least $80 - h$ hours. Robert worked 20 hours his first week with the company. From then on, he always worked the minimum number of hours required of him. Describe the number of hours Robert worked from week to week. Robert alternated 20-hour weeks with 60-hour weeks.

4. GEOMETRY A sequence of triangular shapes is made using squares as shown in the figure.

![Figure 1](image1)

![Figure 2](image2)

Let $x_n$ be the number of squares to make the $n$th figure. Write a recursive formula for $x_n$.

$$x_1 = 1; x_{n+1} = x_n + 2n + 1$$

for $n > 0$

PATHS For Exercises 5 and 6, use the following information.

Gregory wants to know how many different paths he can make of a fixed length. Let $a_n$ denote the number of paths he can make of length $n$ yards.

5. What are the first 5 values of $a_n$?

1, 2, 3, 5, 8

6. Write a recursive formula for $a_n$.

Explain.

$a_1 = 1; a_2 = 2; a_{n+1} = a_n + a_{n-1}$

for $n > 1$. Every path of length $n - 1$ can be extended to a path of length $n + 1$ by adding a 1 by 2 rectangle and every path of length $n$ can be extended to a path of length $n + 1$ by adding a 1 by 1 rectangle.

11-6 Enrichment

Continued Fractions

The fraction below is an example of a continued fraction. Note that each fraction in the continued fraction has a numerator of 1.

$$\frac{2}{3 + \frac{1}{4 + \frac{1}{5 + \frac{1}{6}}}}$$

Example 1 Evaluate the continued fraction above. Start at the bottom and work your way up.

Step 1: $4 + \frac{1}{5} = \frac{21}{5} + \frac{1}{6} = \frac{61}{5}$

Step 2: $\frac{21}{5} + \frac{1}{6} = \frac{21}{6} = \frac{68}{21}$

Step 3: $3 + \frac{68}{21} = \frac{21}{21} + \frac{68}{21} = \frac{89}{21}$

Step 4: $\frac{21}{21} + \frac{89}{21} = \frac{110}{21}$

Step 5: $\frac{110}{21} + \frac{1}{2} = \frac{110 + 21}{42} = \frac{131}{42}$

This fraction can be written as $\frac{11}{42}$.

Example 2 Change $\frac{25}{11}$ into a continued fraction.

Follow the steps.

Step 1: $\frac{25}{11} + \frac{3}{7} + \frac{1}{2} = 2 + \frac{1}{3}$

Step 2: $\frac{11}{3} + \frac{1}{2} = \frac{23}{6}$

Step 3: $\frac{1}{2} + \frac{1}{2} = \frac{1}{2}$

Step 4: $\frac{5}{2} + \frac{1}{2} = \frac{1}{2}$

Step 5: $\frac{3}{2} + \frac{1}{2} = \frac{1}{2}$

Thus, $\frac{25}{11}$ can be written as $2 + \frac{1}{3}$.

Evaluate each continued fraction.

1. $1 + \frac{1}{1 + \frac{1}{2 + \frac{1}{3 + \frac{1}{4}}}} = \frac{17}{24}$

2. $0 + \frac{1}{6 + \frac{1}{4 + \frac{1}{3 + \frac{1}{2}}}} = \frac{9}{56}$

3. $\frac{3}{4} + \frac{1}{6 + \frac{1}{8 + \frac{1}{5}}} = \frac{496}{2065}$

4. $3 + \frac{1}{7 + \frac{1}{9 + \frac{1}{10}}} = \frac{100}{711}$

5. $\frac{7}{8} + \frac{1}{2 + \frac{1}{3 + \frac{1}{4 + \frac{1}{5}}}} = \frac{7}{13}$

6. $\frac{29}{28} + \frac{1}{3 + \frac{1}{2 + \frac{1}{1 + \frac{1}{1}}}} = \frac{13}{19}$

7. $0 + \frac{1}{1 + \frac{1}{2 + \frac{1}{1 + \frac{1}{2}}}} = \frac{13}{19}$

Change each fraction into a continued fraction.

1. $\frac{75}{56}$
2. $\frac{29}{28}$
3. $\frac{13}{19}$
Chapter 11

11-6 Graphing Calculator Activity

Recursion and Iteration

A graphing calculator can be used to perform iterations and recursions.

**Example 1**

Find the first 3 iterates of \( f(x) = 4x + 15 \) if \( x_0 = 5 \).

Store \( x_0 \) in \( X \). Then enter the expression on the home screen. Store the result to \( X \). Repeat the calculation for each iterate.

Keystrokes: \( 5 \) \( \text{STO} \) \( X \) \( \text{STO} \) \( 4 \) \( X \) \( \text{MATH} \) \( 1 \) \( \text{X}^2 \) \( \text{STO} \) \( X \)

\( x_1 = 35 \), \( x_2 = 155 \), and \( x_3 = 635 \)

**Example 2**

A savings account has an initial balance of $2000.00. At the end of each year, the bank pays 6% interest and charges a $20 annual fee. Find the account balance after 6 years.

Store the initial value and enter an expression to calculate the balance at the end of a year.

Keystrokes: \( 2000 \) \( \text{STO} \) \( X \) \( \text{STO} \) \( 1.06 \) \( X \) \( \text{MATH} \) \( 2 \) \( \text{STO} \) \( X \) \( \text{STO} \) \( -20 \) \( \text{STO} \) \( X \)

At the end of six years, the account has a balance of $4116.05.

**Exercises**

Find the first three iterates of each function.

1. \( f(x) = 6x + 12 \) if \( x_0 = 5 \)
   \( x_1 = 42 \), \( x_2 = 264 \), \( x_3 = 1596 \)
2. \( f(x) = 2x^2 - 3 \) if \( x_0 = -1 \)
   \( x_1 = -1.2 \), \( x_2 = -2.5 \)
3. \( f(x) = x^2 - 4x + 5 \) if \( x_0 = 1 \)
   \( x_1 = 0 \), \( x_2 = 1 \), \( x_3 = 2 \)
4. \( f(x) = 3x + 1 \) if \( x_0 = 1 \)
   \( x_1 = 0.3 \), \( x_2 = 0.9 \), \( x_3 = 1.5 \)

A bank account has an initial balance of $11,250.00. Interest is paid at the end of each year. Find the account balance under the given interest rate after the stated time period.

- **5.3%**, 2 years: $11,121.25
- **4.75%**, 5 years: $14,188.05
- **6.05%**, 10 years: $20,242.27
- **7.44%**, 15 years: $33,009.77

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Lesson 11-7

Lesson Reading Guide

The Binomial Theorem

Get Ready for the Lesson

Read the introduction to Lesson 11-7 in your textbook.

- If a family has four children, list the sequences of births of girls and boys that result in three girls and one boy. \( BGGG \), GBGG, GGGB, GGBG, GBGB, GGBB
- Describe a way to figure out how many such sequences there are without listing them.

Sample answer: The boy could be the first, second, third, or fourth child, so there are four sequences with three girls and one boy.

Read the Lesson

1. Consider the expansion of \((u + z)^5\).
   a. How many terms does this expansion have? 6
   b. In the second term of the expansion, what is the exponent of \( u \)? 4
   c. What is the exponent of \( z^4 \)? 1
   d. What is the coefficient of the second term? 5
   e. In the fourth term of the expansion, what is the exponent of \( u^2 \)? 2
   f. What is the exponent of \( z^3 \)? 3
   g. What is the coefficient of the fourth term? 10
   h. What is the last term of this expansion? \( z^5 \)

2. a. State the definition of a factorial in your own words. (Do not use mathematical symbols in your definition.) Sample answer: The factorial of any positive integer is the product of that integer and all the smaller integers down to one. The factorial of zero is one.
   b. Write out the product that you would use to calculate \( 10! \). (Do not actually calculate the product.) \( 10 \cdot 9 \cdot 8 \cdot 7 \cdot 6 \cdot 5 \cdot 4 \cdot 3 \cdot 2 \cdot 1 \)
   c. Write an expression involving factorials that could be used to find the coefficient of the third term of the expansion of \((m - n)^5\). (Do not actually calculate the coefficient.) \( \frac{6!}{4!2!} \)

Remember What You Learned

3. Without using Pascal’s triangle or factorials, what is an easy way to remember the first two and last two coefficients for the terms of the binomial expansion of \((a + b)^7\)?

Sample answer: The first and last coefficients are always 1. The second and next-to-last coefficients are always \( n \), the power to which the binomial is being raised.
11-7 Study Guide and Intervention

The Binomial Theorem

Pascal's Triangle  Pascal's triangle is the pattern of coefficients of powers of binomials displayed in triangular form. Each row begins and ends with 1 and each coefficient is the sum of the two coefficients above it in the previous row.

```
  n = 0
  1
  n = 1
  1 1
  n = 2
  1 2 1
  n = 3
  1 3 3 1
  n = 4
  1 4 6 4 1
  n = 5
  1 5 10 10 5 1
```

Example  Use Pascal's triangle to find the number of possible sequences consisting of 3 as and 2 bs.

The coefficient 10 of the $a^3b^2$ term in the expansion of $(a + b)^5$ gives the number of sequences that result in three as and two bs.

Exercises

Expand each power using Pascal's triangle.

1. $(a + b)^4$  $a^4 + 20a^3b + 150a^2b^2 + 500ab^3 + 625b^4$

2. $(x - 2y)^6$  $x^6 - 12x^5y + 60x^4y^2 - 160x^3y^3 + 240x^2y^4 - 192xy^5 + 64y^6$

3. $(j - 3k)^4$  $j^4 - 15j^3k + 90j^2k^2 - 270jk^3 + 405k^4 - 243k^4$

4. $(2s + t)^5$  $128s^5 + 448s^4t + 672s^3t^2 + 560s^2t^3 + 280st^4 + 84t^5 + 1t^7$

5. $(2p - 3q)^6$  $64p^6 + 576p^5q + 2160p^4q^2 + 4320p^3q^3 + 4860p^2q^4 + 2916pq^5 + 729q^6$

6. $(a - b)^5$  $a^5 - 2a^4b + 3a^3b^2 - \frac{1}{2}a^2b^3 + \frac{1}{16}ab^4$

7. Ray tosses a coin 15 times. How many different sequences of tosses could result in 4 heads and 11 tails? 1365

8. There are 9 true-false questions on a quiz. If twice as many of the statements are true as false, how many different sequences of true-false answers are possible? 84

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11-7 Study Guide and Intervention (continued)

The Binomial Theorem

If $n$ is a nonnegative integer, then

$$(a + b)^n = \binom{n}{0}a^n + \binom{n}{1}a^{n-1}b + \binom{n}{2}a^{n-2}b^2 + \ldots + \binom{n}{n-2}a^2b^{n-2} + \binom{n}{n-1}ab^{n-1} + \binom{n}{n}b^n$$

Another useful form of the Binomial Theorem uses factorial notation and sigma notation.

$$
\sum_{k=0}^{n} \binom{n}{k} a^{n-k} b^k
$$

Example 1  Evaluate $\frac{11!}{8!}$.

$$
\frac{11!}{8!} = 11 \cdot 10 \cdot 9 \cdot 8 \cdot 7 \cdot 6 \cdot 5 \cdot 4 \cdot 3 \cdot 2 \cdot 1
\]

$$
= 11 \cdot 10 \cdot 9 = 990
$$

Example 2  Expand $(a - 3b)^4$.

$$
(a - 3b)^4 = \binom{4}{0}a^4 + \binom{4}{1}a^3(-3b) + \binom{4}{2}a^2(-3b)^2 + \binom{4}{3}a(-3b)^3 + \binom{4}{4}(-3b)^4
\]

$$
= a^4 - 12a^3b + 54a^2b^2 - 108ab^3 + 81b^4
$$

Exercises

Evaluate each expression.

1. $5! = 120$  
2. $\binom{9}{3} = 84$  
3. $\binom{10}{1} = 10$

Expand each power.

4. $(a - 3b)^2 = a^2 - 6ab + 9b^2$  
5. $(r + 2s)^3 = r^3 + 12rs^2 + 84r^2s^3 + 280rs^3 + 560r^2s^3 + 672r^2s^3 + 448rs^5 + 128a^7$

6. $(4s + y)^4 = 256s^4 + 1024s^3y + 1536s^2y^2 + 1152sy^3 + 256y^4$

7. $\left(\frac{a}{b}\right)^3 = 32 - 40m + 20m^2 - 5m^3 + \frac{5}{8}m^4 - \frac{1}{16}m^5$

Find the indicated term of each expansion.

8. third term of $(3x - y)^5$  
9. fifth term of $(a + 1)^6$  
10. fourth term of $(j + 2k)^4$  
11. sixth term of $(10 - 2i)^7$  
12. second term of $(m + \frac{2}{3})^6$  
13. seventh term of $(5x - 2)^11$
11-7 Skills Practice
The Binomial Theorem

Evaluate each expression.
1. 8! = 40,320
2. 10! = 3,628,800
3. 12! = 479,001,600
4. 15! = 210
5. \( \frac{5!}{3!} = 40 \)
6. \( \frac{6!}{2!3!} = 60 \)
7. \( \frac{7!}{4!3!} = 7 \)
8. \( \frac{8!}{6!2!} = 28 \)

Expand each power.
9. \((x - y)^3 = x^3 - 3xy + 3y^2 - y^3\)
10. \((a + b)^3 = a^3 + 3a^2b + 3ab^2 + b^3\)
11. \((x - h)^4 = x^4 - 4hx^3 + 6h^2x^2 - 4h^3x + h^4\)
12. \((m + 1)^4 = m^4 + 4m^3 + 6m^2 + 4m + 1\)
13. \((r + 3)^4 = r^4 + 12r^3 + 54r^2 + 108r + 81\)
14. \((a - 5)^4 = a^4 - 20a^3 + 150a^2 - 500a + 625\)
15. \((y - 7)^5 = y^5 - 35y^4 + 210y^3 - 420y^2 + 350y - 56\)
16. \((d + 2)^5 = d^5 + 10d^4 + 40d^3 + 80d^2 + 80d + 32\)
17. \((x - 1)^6 = x^6 - 6x^5 + 15x^4 - 20x^3 + 15x^2 - 6x + 1\)
18. \((2a - b)^6 = 64a^6 - 192a^5b + 240a^4b^2 - 160a^3b^3 + 60a^2b^4 - 10ab^5 + b^6\)
19. \((c - d)^7 = c^7 - 7c^6d + 21c^5d^2 - 35c^4d^3 + 35c^3d^4 - 21c^2d^5 + 7cd^6 - d^7\)

Find the indicated term of each expansion.
20. fourth term of \((m + n)^{10} = 120m^7n^3\)
21. seventh term of \((x - y)^{28} = 28x^2y^6\)
22. third term of \((b + 6)^6 = 360b^3\)
23. sixth term of \((a - 2)^3 = -4032a^3\)
24. fifth term of \((2a + 3)^6 = 4860a^2\)
25. second term of \((3x - y)^7 = -5103x^6y\)