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11-1 Skills Practice

Arithmetic Sequences

Find the next four terms of each arithmetic sequence.

1. 7, 11, 15, … **19, 23, 27, 31**

2. −10, −5, 0, … **5, 10, 15, 20**

3. 101, 202, 303, … **404, 505, 606, 707**

4. 15, 7, −1, … **−9, −17, −25, −33**

5. −67, −60, −53, … **−46, −39, −32, −25**

6. −12, −15, −18, … **−21, −24, −27, −30**

Find the first five terms of each arithmetic sequence described.

7. $a_1 = 6, d = 9$ **6, 15, 24, 33, 42**

8. $a_1 = 27, d = 4$ **27, 31, 35, 39, 43**

9. $a_1 = -12, d = 5$ **−12, −7, −2, 3, 8**

10. $a_1 = 93, d = -15$ **93, 78, 63, 48, 33**

11. $a_1 = -64, d = 11$ **−64, −53, −42, −31, −20**

12. $a_1 = -47, d = -20$ **−47, −67, −87, −107, −127**

Find the indicated term of each arithmetic sequence.

13. $a_1 = 2, d = 6, n = 12$ **68**

14. $a_1 = 18, d = 2, n = 8$ **32**

15. $a_1 = 23, d = 5, n = 23$ **133**

16. $a_1 = 15, d = -1, n = 25$ **−9**

17. a_{31} for 34, 38, 42, … **154**

18. a_{42} for 27, 30, 33, … **150**

Complete the statement for each arithmetic sequence.

19. 55 is the _?_th term of 4, 7, 10, … . **18**

20. 163 is the _?_th term of −5, 2, 9, … . **25**

Write an equation for the *n*th term of each arithmetic sequence.

21. 4, 7, 10, 13, … $a_n = 3n + 1$

22. −1, 1, 3, 5, … $a_n = 2n - 3$

23. −1, 3, 7, 11, … $a_n = 4n - 5$

24. 7, 2, −3, −8, … $a_n = -5n + 12$

Find the arithmetic means in each sequence.

25. 6, _?_, _?_, _?_, 38 **14, 22, 30**

26. 63, _?_, _?_, _?_, 147 **84, 105, 126**

NAME ______________________ DATE __________ PERIOD _____

11-1 Practice

Arithmetic Sequences

Find the next four terms of each arithmetic sequence.

1. 5, 8, 11, … **14, 17, 20, 23**

2. −4, −6, −8, … **−10, −12, −14, −16**

3. 100, 93, 86, … **79, 72, 65, 58**

4. −24, −19, −14, … **−9, −4, 1, 6**

5. $\frac{7}{2}, 6, \frac{17}{2}, 11, \ldots$ $\mathbf{\frac{27}{2}, 16, \frac{37}{2}, 21}$

6. 4.8, 4.1, 3.4, … **2.7, 2, 1.3, 0.6**

Find the first five terms of each arithmetic sequence described.

7. $a_1 = 7, d = 7$ **7, 14, 21, 28, 35**

8. $a_1 = -8, d = 2$ **−8, −6, −4, −2, 0**

9. $a_1 = -12, d = -4$ **−12, −16, −20, −24, −28**

10. $a_1 = \frac{1}{2}, d = \frac{1}{2}$ $\mathbf{\frac{1}{2}, 1, \frac{3}{2}, 2, \frac{5}{2}}$

11. $a_1 = -\frac{5}{6}, d = -\frac{1}{3}$ $\mathbf{-\frac{5}{6}, -\frac{7}{6}, -\frac{3}{2}, -\frac{11}{6}, -\frac{13}{6}}$

12. $a_1 = 10.2, d = -5.8$ **10.2, 4.4, −1.4, −7.2, −13**

Find the indicated term of each arithmetic sequence.

13. $a_1 = 5, d = 3, n = 10$ **32**

14. $a_1 = 9, d = 3, n = 29$ **93**

15. a_{18} for −6, −7, −8, … . **−23**

16. a_{37} for 124, 119, 114, … . **−56**

17. $a_1 = \frac{9}{5}, d = -\frac{3}{5}, n = 10$ $\mathbf{-\frac{18}{5}}$

18. $a_1 = 14.25, d = 0.15, n = 31$ **18.75**

Complete the statement for each arithmetic sequence.

19. 166 is the _?_th term of 30, 34, 38, … **35**

20. 2 is the _?_th term of $\frac{3}{5}, \frac{4}{5}, 1, \ldots$ **8**

Write an equation for the *n*th term of each arithmetic sequence.

21. −5, −3, −1, 1, … $a_n = 2n - 7$

22. −8, −11, −14, −17, … $a_n = -3n - 5$

23. 1, −1, −3, −5, … $a_n = -2n + 3$

24. −5, 3, 11, 19, … $a_n = 8n - 13$

Find the arithmetic means in each sequence.

25. −5, _?_, _?_, _?_, 11 **−1, 3, 7**

26. 82, _?_, _?_, _?_, 18 **66, 50, 34**

27. EDUCATION Trevor Koba has opened an English Language School in Isehara, Japan. He began with 26 students. If he enrolls 3 new students each week, in how many weeks will he have 101 students? **26 wk**

28. SALARIES Yolanda interviewed for a job that promised her a starting salary of $32,000 with a $1250 raise at the end of each year. What will her salary be during her sixth year if she accepts the job? **$38,250**

Lesson 11-1

Answers

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11-1 Word Problem Practice

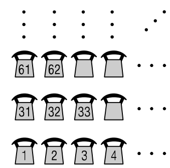
Arithmetic Sequences

1. **ALLOWANCES** Mark has saved \$370 for a scooter and continues to save his weekly allowance of \$10. Find the amount Mark will have saved after 7 weeks.
\$440

2. **GRAPHS** A financial officer is making a graph of a company's financial performance for the month. The vertical axis is labeled "Monthly Profit." The values range from 5400 to 6900. There is not enough space along the vertical axis to write all the numbers between 5400 and 6900, so the financial officer decides to write only 7 numbers, evenly spaced, starting at 5400 and ending at 6900. What should the numbers along the vertical axis be?
5400, 5650, 5900, 6150, 6400, 6650, 6900

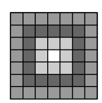
3. **BIKING** City planners want to mark a bike trail with posts that give the distance along the trail to City Hall. The trail begins 37.2 miles from City Hall and ends at City Hall. Write a formula for the number of miles on the nth post if posts are placed every half mile starting at 37.2 miles and decreasing along the way to City Hall.
37.7 – 0.5*n*

4. **SEATING** Kay is trying to find her seat in a theater. The seats are numbered sequentially going left to right. Each row has 30 seats.



The figure shows some of the chairs in the left corner near the stage. Kay is at seat 129, but she needs to find seat 219. She notices that the seat numbers in a fixed column form an arithmetic sequence. What are the numbers of the next 4 seats in the same column as seat 129 going away from the stage? Where does Kay have to go to find her seat? In what row and column is her seat?
159, 189, 219, 249; Kay can move 3 rows back; row 8, column 9

RINGS For Exercises 5-7, use the figure of expanding square rings.



5. How many small squares are in the first few square rings in the figure?
8, 16, 24

6. If the pattern is continued, write a formula for the number of squares in the nth ring.
8*n*

7. What is the side length of the nth ring?
2*n* + 1

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11-1 Enrichment

Fibonacci Sequence

Leonardo Fibonacci first discovered the sequence of numbers named for him while studying rabbits. He wanted to know how many pairs of rabbits would be produced in n months, starting with a single pair of newborn rabbits. He made the following assumptions.

1. Newborn rabbits become adults in one month.
2. Each pair of rabbits produces one pair each month.
3. No rabbits die.

Let F_n represent the number of pairs of rabbits at the end of n months. If you begin with one pair of newborn rabbits, $F_0 = F_1 = 1$. This pair of rabbits would produce one pair at the end of the second month, so $F_2 = 1 + 1$, or 2. At the end of the third month, the first pair of rabbits would produce another pair. Thus, $F_3 = 2 + 1$, or 3.

The chart below shows the number of rabbits each month for several months.

Month	Adult Pairs	Newborn Pairs	Total
F_0	0	1	1
F_1	1	0	1
F_2	1	1	2
F_3	2	1	3
F_4	3	2	5
F_5	5	3	8

Exercises

Solve.

1. Starting with a single pair of newborn rabbits, how many pairs of rabbits would there be at the end of 12 months?
233

2. Write the first 10 terms of the sequence for which $F_0 = 3$, $F_1 = 4$, and $F_n = F_{n-2} + F_{n-1}$.
3, 4, 7, 11, 18, 29, 47, 76, 123, 199, 322

3. Write the first 10 terms of the sequence for which $F_0 = 1$, $F_1 = 5$, $F_n = F_{n-2} + F_{n-1}$.
1, 5, 6, 11, 17, 28, 45, 73, 118, 191, 309

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11-2 Lesson Reading Guide

Arithmetic Series

Get Ready for the Lesson

Read the introduction to Lesson 11-2 in your textbook.

Suppose that an amphitheater can seat 50 people in the first row and that each row thereafter can seat 9 more people than the previous row. Using the vocabulary of arithmetic sequences, describe how you would find the number of people who could be seated in the first 10 rows. (Do not actually calculate the sum.) **Sample answer: Find the first 10 terms of an arithmetic sequence with first term 50 and common difference 9. Then add these 10 terms.**

Read the Lesson

1. What is the relationship between an arithmetic sequence and the corresponding arithmetic series? **Sample answer: An arithmetic sequence is a list of terms with a common difference between successive terms. The corresponding arithmetic series is the sum of the terms of the sequence.**

2. Consider the formula $S_n = \frac{n}{2}(a_1 + a_n)$. Explain the meaning of this formula in words.
Sample answer: To find the sum of the first *n* terms of an arithmetic sequence, find half the number of terms you are adding. Multiply this number by the sum of the first term and the *n*th term.

3. a. What is the purpose of sigma notation?
Sample answer: to write a series in a concise form

b. Consider the expression $\sum_{i=2}^{12}(4i - 2)$.

This form of writing a sum is called **sigma notation**.

The variable i is called the **index of summation**.

The first value of i is **2**.

The last value of i is **12**.

How would you read this expression? **The sum of 4*i* −2 as *i* goes from 2 to 12.**

Remember What You Learned

4. A good way to remember something is to relate it to something you already know. How can your knowledge of how to find the average of two numbers help you remember the formula $S_n = \frac{n}{2}(a_1 + a_n)$? **Sample answer: Rewrite the formula as $S_n = n \cdot \frac{a_1 + a_n}{2}$. The average of the first and last terms is given by the expression $\frac{a_1 + a_n}{2}$. The sum of the first *n* terms is the average of the first and last terms multiplied by the number of terms.**

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11-2 Study Guide and Intervention

Arithmetic Series

Arithmetic Series An **arithmetic series** is the sum of consecutive terms of an arithmetic sequence.

Sum of an Arithmetic Series	The sum S_n of the first n terms of an arithmetic series is given by the formula $S_n = \frac{n}{2}[2a_1 + (n - 1)d]$ or $S_n = \frac{n}{2}(a_1 + a_n)$

Example 1 **Find S_n for the arithmetic series with $a_1 = 14$, $a_n = 101$, and $n = 30$.**

Use the sum formula for an arithmetic series.

$S_n = \frac{n}{2}(a_1 + a_n)$ Sum formula

$S_{30} = \frac{30}{2}(14 + 101)$ $n = 30$, $a_1 = 14$, $a_n = 101$

$= 15(115)$ Simplify.

$= 1725$ Multiply.

The sum of the series is 1725.

Example 2 **Find the sum of all positive odd integers less than 180.**

The series is 1 + 3 + 5 + … + 179.

Find n using the formula for the nth term of an arithmetic sequence.

$a_n = a_1 + (n - 1)d$ Formula for nth term

$179 = 1 + (n - 1)2$ $a_n = 179$, $a_1 = 1$, $d = 2$

$179 = 2n - 1$ Simplify.

$180 = 2n$ Add 1 to each side.

$n = 90$ Divide each side by 2.

Then use the sum formula for an arithmetic series.

$S_n = \frac{n}{2}(a_1 + a_n)$ Sum formula

$S_{90} = \frac{90}{2}(1 + 179)$ $n = 90$, $a_1 = 1$, $a_n = 179$

$= 45(180)$ Simplify.

$= 8100$ Multiply.

The sum of all positive odd integers less than 180 is 8100.

Exercises

Find S_n for each arithmetic series described.

1. $a_1 = 12$, $a_n = 100$, $n = 12$ **672**
2. $a_1 = 50$, $a_n = -50$, $n = 15$ **0**
3. $a_1 = 60$, $a_n = -136$, $n = 50$ **−1900**
4. $a_1 = 20$, $d = 4$, $a_n = 112$ **1584**
5. $a_1 = 180$, $d = -8$, $a_n = 68$ **1860**
6. $a_1 = -8$, $d = -7$, $a_n = -71$ **−395**
7. $a_1 = 42$, $n = 8$, $d = 6$ **504**
8. $a_1 = 4$, $n = 20$, $d = 2\frac{1}{2}$ **555**
9. $a_1 = 32$, $n = 27$, $d = 3$ **1917**

Find the sum of each arithmetic series.

10. 8 + 6 + 4 + … + −10 **−10**
11. 16 + 22 + 28 + … + 112 **1088**

Find the first three terms of each arithmetic series described.

13. $a_1 = 12$, $a_n = 174$, $S_n = 1767$ **12, 21, 30**
14. $a_1 = 80$, $a_n = -115$, $S_n = -245$ **80, 65, 50**
15. $a_1 = 6.2$, $a_n = 12.6$, $S_n = 84.6$ **6.2, 7.0, 7.8**

Lesson 11-2

Answers

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11-2 Study Guide and Intervention (continued)

Arithmetic Series

Sigma Notation A shorthand notation for representing a series makes use of the Greek letter Σ. The **sigma notation** for the series 6 + 12 + 18 + 24 + 30 is $\sum_{n=1}^{5} 6n$.

Example **Evaluate $\sum_{k=1}^{18} (3k + 4)$.**

The sum is an arithmetic series with common difference 3. Substituting $k = 1$ and $k = 18$ into the expression $3k + 4$ gives $a_1 = 3(1) + 4 = 7$ and $a_{18} = 3(18) + 4 = 58$. There are 18 terms in the series, so $n = 18$. Use the formula for the sum of an arithmetic series.

$S_n = \frac{n}{2}(a_1 + a_n)$ Sum formula

$S_{18} = \frac{18}{2}(7 + 58)$ $n = 18$, $a_1 = 7$, $a_n = 58$

$= 9(65)$ Simplify.

$= 585$ Multiply.

So $\sum_{k=1}^{18} (3k + 4) = 585$.

Exercises

Find the sum of each arithmetic series.

1. $\sum_{n=1}^{20} (2n + 1)$ **440**
2. $\sum_{n=5}^{25} (x - 1)$ **294**
3. $\sum_{k=1}^{18} (2k - 7)$ **216**
4. $\sum_{r=10}^{75} (2r - 200)$ **−7590**
5. $\sum_{x=1}^{15} (6x + 3)$ **765**
6. $\sum_{t=1}^{50} (500 - 6t)$ **17,350**
7. $\sum_{k=1}^{80} (100 - k)$ **4760**
8. $\sum_{n=20}^{85} (n - 100)$ **−3135**
9. $\sum_{s=1}^{200} 3s$ **60,300**
10. $\sum_{m=14}^{28} (2m - 50)$ **−120**
11. $\sum_{p=1}^{36} (5p - 20)$ **2610**
12. $\sum_{j=12}^{32} (25 - 2j)$ **−399**
13. $\sum_{n=18}^{42} (4n - 9)$ **2775**
14. $\sum_{n=20}^{50} (3n + 4)$ **3379**
15. $\sum_{j=5}^{44} (7j - 3)$ **6740**

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11-2 Skills Practice

Arithmetic Series

Find S_n for each arithmetic series described.

1. $a_1 = 1, a_n = 19, n = 10$ **100**
2. $a_1 = -5, a_n = 13, n = 7$ **28**
3. $a_1 = 12, a_n = -23, n = 8$ **−44**
4. $a_1 = 7, n = 11, a_n = 67$ **407**
5. $a_1 = 5, n = 10, a_n = 32$ **185**
6. $a_1 = -4, n = 10, a_n = -22$ **−130**
7. $a_1 = -8, d = -5, n = 12$ **−426**
8. $a_1 = 1, d = 3, n = 15$ **330**
9. $a_1 = 100, d = -7, a_n = 37$ **685**
10. $a_1 = -9, d = 4, a_n = 27$ **90**
11. $d = 2, n = 26, a_n = 42$ **442**
12. $d = -12, n = 11, a_n = -52$ **88**

Find the sum of each arithmetic series.

13. $1 + 4 + 7 + 10 + \ldots + 43$ **330**
14. $5 + 8 + 11 + 14 + \ldots + 32$ **185**
15. $3 + 5 + 7 + 9 + \ldots + 19$ **99**
16. $-2 + (-5) + (-8) + \ldots + (-20)$ **−77**
17. $\sum_{n=1}^{5} (2n - 3)$ **15**
18. $\sum_{n=1}^{18} (10 + 3n)$ **693**
19. $\sum_{n=2}^{10} (4n + 1)$ **225**
20. $\sum_{n=5}^{12} (4 - 3n)$ **−172**

Find the first three terms of each arithmetic series described.

21. $a_1 = 4, a_n = 31, S_n = 175$ **4, 7, 10**
22. $a_1 = -3, a_n = 41, S_n = 228$ **−3, 1, 5**
23. $n = 10, a_n = 41, S_n = 230$ **5, 9, 13**
24. $n = 19, a_n = 85, S_n = 760$ **−5, 0, 5**

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11-2 Practice

Arithmetic Series

Find S_n for each arithmetic series described.

1. $a_1 = 16$, $a_n = 98$, $n = 13$ **741**
2. $a_1 = 3$, $a_n = 36$, $n = 12$ **234**
3. $a_1 = -5$, $a_n = -26$, $n = 8$ **−124**
4. $a_1 = 5$, $n = 10$, $a_n = -13$ **−40**
5. $a_1 = 6$, $n = 15$, $a_n = -22$ **−120**
6. $a_1 = -20$, $n = 25$, $a_n = 148$ **1600**
7. $a_1 = 13$, $d = -6$, $n = 21$ **−987**
8. $a_1 = 5$, $d = 4$, $n = 11$ **275**
9. $a_1 = 5$, $d = 2$, $a_n = 33$ **285**
10. $a_1 = -121$, $d = 3$, $a_n = 5$ **−2494**
11. $d = 0.4$, $n = 10$, $a_n = 3.8$ **20**
12. $d = -\frac{2}{3}$, $n = 16$, $a_n = 44$ **784**

Find the sum of each arithmetic series.

13. $5 + 7 + 9 + 11 + \ldots + 27$ **192**
14. $-4 + 1 + 6 + 11 + \ldots + 91$ **870**
15. $13 + 20 + 27 + \ldots + 272$ **5415**
16. $89 + 86 + 83 + 80 + \ldots + 20$ **1308**
17. $\sum_{n=1}^{4}(1 - 2n)$ **−16**
18. $\sum_{j=1}^{6}(5 + 3n)$ **93**
19. $\sum_{n=1}^{5}(9 - 4n)$ **−15**
20. $\sum_{k=4}^{10}(2k + 1)$ **105**
21. $\sum_{n=3}^{8}(5n - 10)$ **105**
22. $\sum_{n=1}^{101}(4 - 4n)$ **−20,200**

Find the first three terms of each arithmetic series described.

23. $a_1 = 14$, $a_n = -85$, $S_n = -1207$ **14, 11, 8**
24. $a_1 = 1$, $a_n = 19$, $S_n = 100$ **1, 3, 5**
25. $n = 16$, $a_n = 15$, $S_n = -120$ **−30, −27, −24**
26. $n = 15$, $a_n = 5\frac{4}{5}$, $S_n = 45$ **$\frac{1}{5}, \frac{3}{5}, 1$**

27. **STACKING** A health club rolls its towels and stacks them in layers on a shelf. Each layer of towels has one less towel than the layer below it. If there are 20 towels on the bottom layer and one towel on the top layer, how many towels are stacked on the shelf? **210 towels**

28. **BUSINESS** A merchant places \$1 in a jackpot on August 1, then draws the name of a regular customer. If the customer is present, he or she wins the \$1 in the jackpot. If the customer is not present, the merchant adds \$2 to the jackpot on August 2 and draws another name. Each day the merchant adds an amount equal to the day of the month. If the first person to win the jackpot wins \$496, on what day of the month was her or his name drawn? **August 31**

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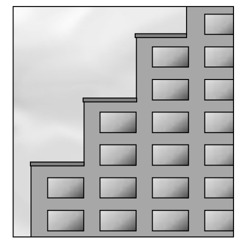
11-2 Word Problem Practice

Arithmetic Series

1. **WINDOWS** A side of an apartment building is shaped like a steep staircase. The windows are arranged in columns. The first column has 2 windows, the next has 4, then 6, and so on. How many windows are on the side of the apartment building if it has 15 columns?



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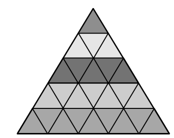
2. **WEIGHTS** Nathan has a collection of barbells for his home gym. He has 2 barbells for every 5 pounds starting at 5 pounds and going up to 80 pounds. What is the total weight of all his barbells? **1,360 lb**

3. **TRAINING** Matthew is training to run a marathon. He runs 20 miles his first week of training. Each week, he increases the number of miles he runs by 4 miles. How many total miles did he run in 8 weeks of training? **272 mi**

4. **VOLUNTEERING** Maryland Public Schools requires all high school students to complete 75 hours of volunteer service as a condition for graduation. One school includes grades 1-12, with 50 students in each grade. The school decides that students in grade g will volunteer $0.25g$ hours per week of their time. How many hours will all the school's students collectively donate to charity each week? **975 hours**

TRIANGLES For Exercises 5-7, use the following information.



A triangle is made of congruent equilateral triangles as shown in the figure.

5. Starting from the top, each colored row of triangles has more and more triangles. Write a formula for the number of triangles in row n. **$2n - 1$**

6. If the large triangle consists of N rows of small triangles, how many small triangles are there in the large triangle? Write your answer using sigma notation. **$\sum_{n=1}^{N}(2n - 1)$**

7. Evaluate the sum you wrote for Exercise 6. **N^2**

Lesson 11-2

Answers

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11-2 Enrichment

Arithmetic Series in Computer Programming

Arithmetic series are used in the analysis of the efficiency of computer programs. Computers effortlessly automate time consuming, often repetitive tasks such as addition and multiplication of numbers. These repetitive tasks are carried out using a *Loop* statement provided by a programming language to execute the calculations until a logical condition is, or is not, satisfied. The loop usually repeats a calculation followed by an assignment statement, which is assigning the number to a specific memory location in the computer.

Suppose you were writing a program to calculate the sum of the numbers from 1 to 10, that is $1 + 2 + 3 + 4 + 5 + 6 + 7 + 8 + 9 + 10$. Two algorithms to calculate this series are shown in the table with the sequential step of the algorithm in the left column.

Step Number	Algorithm
1	Assign in memory s = 1
2	Assign j = 2
3	If j < 11 then do steps 4 and 5
4	Assign s = s + j
5	Assign j = j + 1

1. Write an algorithm segment in pseudo-code (like in the table) which for any given values of a, d, and n—the initial value, the common difference, and the number of terms in the progression, respectively—computes the sum of the series, $\sum_{i=0}^{n} a + id$.

sum = a
for i = 1 to n
sum = a + i * d
next i

2. Double summations are used to analyze nested loops (loops inside of loops). Calculate the double sums below. Start with the inner summation first and then proceed to the outer summation.
$\sum_{i=1}^{4}\sum_{j=1}^{3} ij = \sum_{i=1}^{4}(i + 2i + 3i) = \sum_{i=1}^{4} 6i = 6 + 12 + 18 + 24 = 60.$

Also recall the sum of a arithmetic series is equal to $\frac{n}{2}(a_1 + a_n)$, where n is the number of terms in the series, a_1 is the first term of the sequence and a_n is the last term.

a). $\sum_{i=1}^{2}\sum_{j=1}^{2}(2i + 3j)$ **30**   b). $\sum_{i=1}^{n}\sum_{j=1}^{m} ij$ $\boldsymbol{\frac{n}{2} \times (n + 1) \times \left(\frac{m^2 + m}{2}\right)}$

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11-2 Spreadsheet Activity

Sequences and Series

You have learned about the characteristics of numbers in a sequence. A spreadsheet can calculate a sequence and enable you to find the sum of terms in the series.

Example **Create a spreadsheet like the one below and enter the first three terms of a sequence. Find the first ten terms of the sequence. Then find the sum of the first ten terms of the series.**

◇	A	B	C	D	E	F	E	F	G	H	I	J	K
1	Symbol	a1	a2	a3									
2	Term	3	2.5	2									
3													

Sheet 1 / Sheet 2 / Sheet 3

Highlight cells B2 through D2 and move your cursor to any corner of the highlighted cells until a black cross appears. Drag across the row and release it at cell K2. The next values in the sequence will appear in the cells.

To find the sum of the first 10 terms in the series, highlight the cells containing the terms, then click the $\sum$ symbol on the toolbar. The sum will appear in the next cell. Note that this will work for arithmetic series only. The sum of the first ten terms of this series is 7.5.

Exercises

1. Create a spreadsheet like the one in the example above. Record the initial sequence as −4, −1, and 2. Repeat the process you followed in the example. What are the next six numbers in the sequence?
5, 8, 11, 14, 17, and 20

2. Describe the steps the spreadsheet program completes to find the next term in the sequence. **First, the program calculates the common difference by subtracting any term from its succeeding term. Then, it adds the common difference to the last term to find the next term in the sequence.**

3. Use the spreadsheet to find the value for the 16th term in the sequence.
41

4. Find the sum of the 3rd through 13th terms in the sequence. **187**

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11-3 Lesson Reading Guide

Geometric Sequences

Get Ready for the Lesson

Read the introduction to Lesson 11-3 in your textbook.

Suppose that you drop a ball from a height of 4 feet, and that each time it falls, it bounces back to 74% of the height from which it fell. Describe how would you find the height of the third bounce. (Do not actually calculate the height of the bounce.)

Sample answer: Multiply 4 by 0.74 three times.

Read the Lesson

1. Explain the difference between an arithmetic sequence and a geometric sequence.
   Sample answer: In an arithmetic sequence, each term after the first is found by adding the common difference to the previous term. In a geometric sequence, each term after the first is found by multiplying the previous term by the common ratio.

2. Consider the formula $a_n = a_1 \cdot r^{n-1}$.
   a. What is this formula used to find? **a particular term of a geometric sequence**
   b. What do each of the following represent?
      a_n: **the *n*th term**
      a_1: **the first term**
      r: **the common ratio**
      n: **a positive integer that indicates which term you are finding**

3. a. In the sequence 5, 8, 11, 14, 17, 20, the numbers 8, 11, 14, and 17 are **arithmetic means** between 5 and 20.
   b. In the sequence 12, 4, $\frac{4}{3}$, $\frac{4}{9}$, $\frac{4}{27}$, the numbers 4, $\frac{4}{3}$, and $\frac{4}{9}$ are **geometric means** between 12 and $\frac{4}{27}$.

Remember What You Learned

4. Suppose that your classmate Ricardo has trouble remembering the formula $a_n = a_1 \cdot r^{n-1}$ correctly. He thinks that the formula should be $a_n = a_1 \cdot r^n$. How would you explain to him that he should use r^{n-1} rather than r^n in the formula?
   Sample answer: Each term after the first in a geometric sequence is found by multiplying the previous term by *r*. There are *n* – 1 terms before the *n*th term, so you would need to multiply by *r* a total of *n* – 1 times, not *n* times, to get the *n*th term.

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11-3 Study Guide and Intervention

Geometric Sequences

Geometric Sequences A **geometric sequence** is a sequence in which each term after the first is the product of the previous term and a constant called the **constant ratio**.

*n*th Term of a Geometric Sequence	$a_n = a_1 \cdot r^{n-1}$, where a_1 is the first term, r is the common ratio, and n is any positive integer

Example 1 **Find the next two terms of the geometric sequence 1200, 480, 192, … .**

Since $\frac{480}{1200} = 0.4$ and $\frac{192}{480} = 0.4$, the sequence has a common ratio of 0.4. The next two terms in the sequence are $192(0.4) = 76.8$ and $76.8(0.4) = 30.72$.

Example 2 **Write an equation for the *n*th term of the geometric sequence 3.6, 10.8, 32.4, … .**

In this sequence $a_1 = 3.6$ and $r = 3$. Use the nth term formula to write an equation.

$a_n = a_1 \cdot r^{n-1}$ Formula for nth term
$= 3.6 \cdot 3^{n-1}$ $a_1 = 3.6$, $r = 3$

An equation for the nth term is $a_n = 3.6 \cdot 3^{n-1}$.

Exercises

Find the next two terms of each geometric sequence.

1. 6, 12, 24, … **48, 96**
2. 180, 60, 20, … **$\frac{20}{3}, \frac{20}{9}$**
3. 2000, −1000, 500, … **−250, 125**
4. 0.8, −2.4, 7.2, … **−21.6, 64.8**
5. 80, 60, 45, … **33.75, 25.3125**
6. 3, 16.5, 90.75, … **499.125, 2745.1875**

Find the first five terms of each geometric sequence described.

7. $a_1 = \frac{1}{9}$, $r = 3$ **$\frac{1}{9}, \frac{1}{3}, 1, 3, 9$**
8. $a_1 = 240$, $r = -\frac{3}{4}$ **$240, -180, 135, -101\frac{1}{4}, 75\frac{15}{16}$**
9. $a_1 = 10$, $r = \frac{5}{2}$ **$10, 25, 62\frac{1}{2}, 156\frac{1}{4}, 390\frac{5}{8}$**

Find the indicated term of each geometric sequence.

10. $a_1 = -10$, $r = 4$, $n = 2$ **−40**
11. $a_1 = -6$, $r = -\frac{1}{2}$, $n = 8$ **$\frac{3}{64}$**
12. $a_3 = 9$, $r = -3$, $n = 7$ **729**
13. $a_4 = 16$, $r = 2$, $n = 10$ **1024**
14. $a_4 = -54$, $r = -3$, $n = 6$ **−486**
15. $a_1 = 8$, $r = \frac{2}{3}$, $n = 5$ **$\frac{128}{81}$**

Write an equation for the *n*th term of each geometric sequence.

16. 500, 350, 245, … **$500 \cdot 0.7^{n-1}$**
17. 8, 32, 128, … **$8 \cdot 4^{n-1}$**
18. 11, −24.2, 53.24, … **$11 \cdot (-2.2)^{n-1}$**

Lesson 11-3

Answers

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11-3 Study Guide and Intervention *(continued)*

Geometric Sequences

Geometric Means The **geometric means** of a geometric sequence are the terms between any two nonsuccessive terms of the sequence.

To find the k geometric means between two terms of a sequence, use the following steps.

Step 1 Let the two terms given be a_1 and a_n, where $n = k + 2$.
Step 2 Substitute in the formula $a_n = a_1 \cdot r^{n-1}$ $(= a_1 \cdot r^{k+1})$.
Step 3 Solve for r, and use that value to find the k geometric means: $a_1 \cdot r, a_1 \cdot r^2, \ldots, a_1 \cdot r^k$

Example **Find the three geometric means between 8 and 40.5.**

Use the nth term formula to find the value of r. In the sequence 8, _?_, _?_, _?_, 40.5, a_1 is 8 and a_5 is 40.5.

$a_n = a_1 \cdot r^{n-1}$ — Formula for nth term
$40.5 = 8 \cdot r^{5-1}$ — $n = 5$, $a_1 = 8$, $a_5 = 40.5$
$5.0625 = r^4$ — Divide each side by 8.
$r = \pm 1.5$ — Take the fourth root of each side.

There are two possible common ratios, so there are two possible sets of geometric means. Use each value of r to find the geometric means.

$r = 1.5$	$r = -1.5$
$a_2 = 8(1.5)$ or 12	$a_2 = 8(-1.5)$ or -12
$a_3 = 12(1.5)$ or 18	$a_3 = -12(-1.5)$ or 18
$a_4 = 18(1.5)$ or 27	$a_4 = 18(-1.5)$ or -27

The geometric means are 12, 18, and 27, or −12, 18, and −27.

Exercises

Find the geometric means in each sequence.

1. 5, _?_, _?_, _?_, 405
±15, 45, ±135

2. 5, _?_, _?_, 20.48
8, 12.8

3. $\frac{3}{5}$, _?_, _?_, _?_, 375
±3, 15, ±75

4. −24, _?_, _?_, $\frac{1}{9}$
$4, -\frac{2}{3}$

5. 12, _?_, _?_, _?_, _?_, _?_, $\frac{3}{16}$
$\pm 6, 3, \pm\frac{3}{2}, \frac{3}{4}, \pm\frac{3}{8}$

6. 200, _?_, _?_, _?_, 414.72
±240, 288, ±345.6

7. $\frac{35}{49}$, _?_, _?_, _?_, _?_, −12,005
$-\frac{35}{7}$, 35, −245, 1715

8. 4, _?_, _?_, _?_, $156\frac{1}{4}$
$\pm 10, 25, \pm 62\frac{1}{2}$

9. $-\frac{1}{81}$, _?_, _?_, _?_, _?_, _?_, −9
$\pm\frac{1}{27}, -\frac{1}{9}, \pm\frac{1}{3}, -1, \pm 3$

10. 100, _?_, _?_, _?_, 384.16
±140, 196, ±274.4

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11-3 Skills Practice

Geometric Sequences

Find the next two terms of each geometric sequence.

1. −1, −2, −4, … **−8, −16**

2. 6, 3, $\frac{3}{2}$, … **$\frac{3}{4}, \frac{3}{8}$**

3. −5, −15, −45, … **−135, −405**

4. 729, −243, 81, … **−27, 9**

5. 1536, 384, 96, … **24, 6**

6. 64, 160, 400, … **1000, 2500**

Find the first five terms of each geometric sequence described.

7. $a_1 = 6, r = 2$
6, 12, 24, 48, 96

8. $a_1 = -27, r = 3$
−27, −81, −243, −729, −2187

9. $a_1 = -15, r = -1$
−15, 15, −15, 15, −15

10. $a_1 = 3, r = 4$
3, 12, 48, 192, 768

11. $a_1 = 1, r = \frac{1}{2}$
$1, \frac{1}{2}, \frac{1}{4}, \frac{1}{8}, \frac{1}{16}$

12. $a_1 = 216, r = -\frac{1}{3}$
$216, -72, 24, -8, \frac{8}{3}$

Find the indicated term of each geometric sequence.

13. $a_1 = 5, r = 2, n = 6$ **160**

14. $a_1 = 18, r = 3, n = 6$ **4374**

15. $a_1 = -3, r = -2, n = 5$ **−48**

16. $a_1 = -20, r = -2, n = 9$ **−5120**

17. a_8 for −12, −6, −3, … **$-\frac{3}{32}$**

18. a_7 for 80, $\frac{80}{3}$, $\frac{80}{9}$, … **$\frac{80}{729}$**

Write an equation for the *n*th term of each geometric sequence.

19. 3, 9, 27, … **$a_n = 3^n$**

20. −1, −3, −9, … **$a_n = -1(3)^{n-1}$**

21. 2, −6, 18, … **$a_n = 2(-3)^{n-1}$**

22. 5, 10, 20, … **$a_n = 5(2)^{n-1}$**

Find the geometric means in each sequence.

23. 4, _?_, _?_, _?_, 64 **±8, 16, ±32**

24. 1, _?_, _?_, _?_, 81 **±3, 9, ±27**

Lesson 11-3

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11-3 Practice

Geometric Sequences

Find the next two terms of each geometric sequence.

1. $-15, -30, -60, \ldots$ **$-120, -240$**
2. $80, 40, 20, \ldots$ **$10, 5$**
3. $90, 30, 10, \ldots$ **$\frac{10}{3}, \frac{10}{9}$**
4. $-1458, 486, -162, \ldots$ **$54, -18$**
5. $\frac{1}{4}, \frac{3}{8}, \frac{9}{16}, \ldots$ **$\frac{27}{32}, \frac{81}{64}$**
6. $216, 144, 96, \ldots$ **$64, \frac{128}{3}$**

Find the first five terms of each geometric sequence described.

7. $a_1 = -1, r = -3$ **$-1, 3, -9, 27, -81$**
8. $a_1 = 7, r = -4$ **$7, -28, 112, -448, 1792$**
9. $a_1 = -\frac{1}{3}, r = 2$ **$-\frac{1}{3}, -\frac{2}{3}, -\frac{4}{3}, -\frac{8}{3}, -\frac{16}{3}$**
10. $a_1 = 12, r = \frac{2}{3}$ **$12, 8, \frac{16}{3}, \frac{32}{9}, \frac{64}{27}$**

Find the indicated term of each geometric sequence.

11. $a_1 = 5, r = 3, n = 6$ **1215**
12. $a_1 = 20, r = -3, n = 6$ **-4860**
13. $a_1 = -4, r = -2, n = 10$ **2048**
14. a_8 for $-\frac{1}{250}, -\frac{1}{50}, -\frac{1}{10}, \ldots$ **$-\frac{625}{2}$**
15. a_{12} for $96, 48, 24, \ldots$ **$\frac{3}{64}$**
16. $a_1 = 8, r = \frac{1}{2}, n = 9$ **$\frac{1}{32}$**
17. $a_1 = -3125, r = -\frac{1}{5}, n = 9$ **$-\frac{1}{125}$**
18. $a_1 = 3, r = \frac{1}{10}, n = 8$ **$\frac{3}{10{,}000{,}000}$**

Write an equation for the *n*th term of each geometric sequence.

19. $1, 4, 16, \ldots$ **$a_n = (4)^{n-1}$**
20. $-1, -5, -25, \ldots$ **$a_n = -1(5)^{n-1}$**
21. $1, \frac{1}{2}, \frac{1}{4}, \ldots$ **$a_n = \left(\frac{1}{2}\right)^{n-1}$**
22. $-3, -6, -12, \ldots$ **$a_n = -3(2)^{n-1}$**
23. $7, -14, 28, \ldots$ **$a_n = 7(-2)^{n-1}$**
24. $-5, -30, -180, \ldots$ **$a_n = -5(6)^{n-1}$**

Find the geometric means in each sequence.

25. 3, _?_, _?_, _?_, 768 **12, 48, 192**
26. 5, _?_, _?_, _?_, 1280 **±20, 80, ±320**
27. 144, _?_, _?_, _?_, 9 **±72, 36, ±18**
28. 37,500, _?_, _?_, _?_, _?_, −12 **−7500, 1500, −300, 60**

29. **BIOLOGY** A culture initially contains 200 bacteria. If the number of bacteria doubles every 2 hours, how many bacteria will be in the culture at the end of 12 hours? **12,800**

30. **LIGHT** If each foot of water in a lake screens out 60% of the light above, what percent of the light passes through 5 feet of water? **1.024%**

31. **INVESTING** Raul invests $1000 in a savings account that earns 5% interest compounded annually. How much money will he have in the account at the end of 5 years? **$1276.28**

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11-3 Word Problem Practice

Geometric Sequences

1. **INVESTMENT** Beth deposits $1500 into a retirement account that pays an APR of 8% compounded yearly. Assuming Beth makes no withdrawals, how much money will she have in her account after 20 years?
$6991.44

2. **CAKE** Lauren has a piece of cake. She decides she wants to save some for later, so she eats half of it. Each time she returns to what remains, she only eats half of what is left. After her *n*th serving of ever smaller portions of cake, how much of the piece remains?
$(0.5)^n$ of the original piece.

3. **MOORE'S LAW** Gordon Moore, co-founder of Intel, suggested that the number of transistors on a square inch of integrated circuit in a computer chip would double every 18 months. Assuming Moore's law is true, how many times as many transistors would you expect on a square inch of integrated circuit every 18 months for the next 6 years?
2, 4, 8, 16

4. **MONGEESE** A population of mongeese has been growing by 20% every year. If the initial population size was 5000 mongeese, what is the size of the mongoose population after *n* years? How many years will it take, roughly, for the mongoose population to exceed 10,000 mongeese?
$5000(1.2)^n$, 4 years

BIOLOGY For Exercises 5–7, use the following information.

Mitosis is a process of cell division that results in two identical daughter cells from a single parent cell. The table illustrates the number of cells produced after each of the first 5 cell divisions.

Division Number	0	1	2	3	4	5
Number of Cells	1	2	4	8	16	32

5. Do the entries in the "Number of Cells" row form a geometric series? If so, find *r*.
Yes; $r = 2$

6. Write an expression to find the *n*th term of the sequence.
$a_n = 2^{n-1}$

7. Find the number of cells after 100 divisions.
6.34×10^{29}

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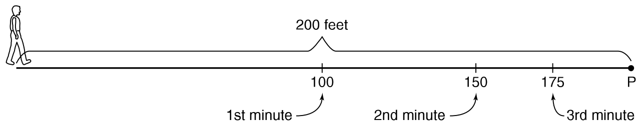
11-3 Enrichment

Half the Distance

Suppose you are 200 feet from a fixed point, P. Suppose that you are able to move to the halfway point in one minute, to the next halfway point one minute after that, and so on.



An interesting sequence results because according to the problem, you never actually reach the point P, although you do get arbitrarily close to it.

You can compute how long it will take to get within some specified small distance of the point. On a calculator, you enter the distance to be covered and then count the number of successive divisions by 2 necessary to get within the desired distance.

Example **How many minutes are needed to get within 0.1 foot of a point 200 feet away?**

Count the number of times you divide by 2.

Enter: 200 [÷] 2 [ENTER] [÷] 2 [ENTER] [÷] 2 [ENTER], and so on

Result: 0.0976562

You divided by 2 eleven times. The time needed is 11 minutes.

Exercises **Use the method illustrated above to solve each problem.**

1. If it is about 2500 miles from Los Angeles to New York, how many minutes would it take to get within 0.1 mile of New York? How far from New York are you at that time? **15 minutes, 0.0762934 mile**

2. If it is 25,000 miles around Earth, how many minutes would it take to get within 0.5 mile of the full distance around Earth? How far short would you be? **16 minutes; 0.3814697 mile**

3. If it is about 250,000 miles from Earth to the Moon, how many minutes would it take to get within 0.5 mile of the Moon? How far from the surface of the Moon would you be? **19 minutes, 0.4768372 mile**

4. If it is about 30,000,000 feet from Honolulu to Miami, how many minutes would it take to get to within 1 foot of Miami? How far from Miami would you be at that time? **25 minutes, 0.8940697 foot**

5. If it is about 93,000,000 miles to the sun, how many minutes would it take to get within 500 miles of the sun? How far from the sun would you be at that time? **18 minutes, 354.766846 miles**

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11-4 Lesson Reading Guide

Geometric Series

Get Ready for the Lesson

Read the introduction to Lesson 11-4 in your textbook.

- Suppose that you e-mail the joke on Monday to five friends, rather than three, and that each of those friends e-mails it to five friends on Tuesday, and so on. Write a sum that shows that total number of people, including yourself, who will have read the joke by Thursday. (Write out the sum using plus signs rather than sigma notation. Do not actually find the sum.) **$1 + 5 + 25 + 125$**
- Use exponents to rewrite the sum you found above. (Use an exponent in each term, and use the same base for all terms.) **$5^0 + 5^1 + 5^2 + 5^3$**

Read the Lesson

1. Consider the formula $S_n = \frac{a_1(1 - r^n)}{1 - r}$.

   a. What is this formula used to find? **the sum of the first n terms of a geometric series**

   b. What do each of the following represent?

   S_n: **the sum of the first n terms**

   a_1: **the first term**

   r: **the common ratio**

   c. Suppose that you want to use the formula to evaluate $3 - 1 + \frac{1}{3} - \frac{1}{9} + \frac{1}{27}$. Indicate the values you would substitute into the formula in order to find S_n. (Do not actually calculate the sum.)

   $n =$ **5** $a_1 =$ **3** $r =$ **$-\frac{1}{3}$** $r^n =$ **$\left(-\frac{1}{3}\right)^5$ or $-\frac{1}{243}$**

   d. Suppose that you want to use the formula to evaluate the sum $\sum_{n=1}^{6} 8(-2)^{n-1}$. Indicate the values you would substitute into the formula in order to find S_n. (Do not actually calculate the sum.)

   $n =$ **6** $a_1 =$ **8** $r =$ **-2** $r^n =$ **$(-2)^6$ or 64**

Remember What You Learned

2. This lesson includes three formulas for the sum of the first n terms of a geometric series. All of these formulas have the same denominator and have the restriction $r \neq 1$. How can this restriction help you to remember the denominator in the formulas? **Sample answer: If $r = 1$, then $r - 1 = 0$. Because division by 0 is undefined, a formula with $r - 1$ in the denominator will not apply when $r = 1$.**

Lesson 11-4

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11-4 Study Guide and Intervention

Geometric Series

Geometric Series A **geometric series** is the indicated sum of consecutive terms of a geometric sequence.

Sum of a Geometric Series	The sum S_n of the first n terms of a geometric series is given by $S_n = \frac{a_1(1 - r^n)}{1 - r}$ or $S_n = \frac{a_1 - a_1r^n}{1 - r}$, where $r \neq 1$.

Example 1 **Find the sum of the first four terms of the geometric sequence for which $a_1 = 120$ and $r = \frac{1}{3}$.**

$S_n = \frac{a_1(1 - r^n)}{1 - r}$ Sum formula

$S_4 = \frac{120\left(1 - \left(\frac{1}{3}\right)^4\right)}{1 - \frac{1}{3}}$ n 4, a_1 120, r $\frac{1}{3}$

≈ 177.78 Use a calculator.

The sum of the series is 177.78.

Example 2 **Find the sum of the geometric series $\sum_{j=1}^{7} 4 \cdot 3^{j-2}$.**

Since the sum is a geometric series, you can use the sum formula.

$S_n = \frac{a_1(1 - r^n)}{1 - r}$ Sum formula

$S_7 = \frac{\frac{4}{3}(1 - 3^7)}{1 - 3}$ n 7, a_1 $\frac{4}{3}$, r 3

≈ 1457.33 Use a calculator.

The sum of the series is 1457.33.

Exercises

Find S_n for each geometric series described.

1. $a_1 = 2, a_n = 486, r = 3$ **728**
2. $a_1 = 1200, a_n = 75, r = \frac{1}{2}$ **2325**
3. $a_1 = \frac{1}{25}, a_n = 125, r = 5$ **156.24**
4. $a_1 = 3, r = \frac{1}{3}, n = 4$ **4.44**
5. $a_1 = 2, r = 6, n = 4$ **518**
6. $a_1 = 2, r = 4, n = 6$ **2730**
7. $a_1 = 100, r = -\frac{1}{2}, n = 5$ **68.75**
8. $a_3 = 20, a_6 = 160, n = 8$ **1275**
9. $a_4 = 16, a_7 = 1024, n = 10$ **87,381.25**

Find the sum of each geometric series.

10. $6 + 18 + 54 + \ldots$ to 6 terms **2184**
11. $\frac{1}{4} + \frac{1}{2} + 1 + \ldots$ to 10 terms **255.75**
12. $\sum_{j=4}^{8} 2^j$ **496**
13. $\sum_{k=1}^{7} 3 \cdot 2^{k-1}$ **381**

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11-4 Study Guide and Intervention *(continued)*

Geometric Series

Specific Terms You can use one of the formulas for the sum of a geometric series to help find a particular term of the series.

Example 1 **Find a_1 in a geometric series for which $S_6 = 441$ and $r = 2$.**

$S_n = \frac{a_1(1 - r^n)}{1 - r}$ Sum formula

$441 = \frac{a_1(1 - 2^6)}{1 - 2}$ S_6 441, r 2, n 6

$441 = \frac{-63a_1}{-1}$ Subtract.

$a_1 = \frac{441}{63}$ Divide.

$a_1 = 7$ Simplify.

The first term of the series is 7.

Example 2 **Find a_1 in a geometric series for which $S_n = 244$, $a_n = 324$, and $r = -3$.**

Since you do not know the value of n, use the alternate sum formula.

$S_n = \frac{a_1 - a_nr}{1 - r}$ Alternate sum formula

$244 = \frac{a_1 - (324)(-3)}{1 - (-3)}$ S_n 244, a_n 324, r −3

$244 = \frac{a_1 + 972}{4}$ Simplify.

$976 = a_1 + 972$ Multiply each side by 4.

$a_1 = 4$ Subtract 972 from each side.

The first term of the series is 4.

Example 3 **Find a_4 in a geometric series for which $S_n = 796.875$, $r = \frac{1}{2}$, and $n = 8$.**

First use the sum formula to find a_1.

$S_n = \frac{a_1(1 - r^n)}{1 - r}$ Sum formula

$796.875 = \frac{a_1\left(1 - \left(\frac{1}{2}\right)^8\right)}{1 - \frac{1}{2}}$ S_8 = 796.875, r = $\frac{1}{2}$, n = 8

$796.875 = \frac{0.99609375a_1}{0.5}$ Use a calculator.

$a_1 = 400$

Since $a_4 = a_1 \cdot r^3$, $a_4 = 400\left(\frac{1}{2}\right)^3 = 50$. The fourth term of the series is 50.

Exercises

Find the indicated term for each geometric series described.

1. $S_n = 726, a_n = 486, r = 3; a_1$ **6**
2. $S_n = 850, a_n = 1280, r = -2; a_1$ **−10**
3. $S_n = 1023.75, a_n = 512, r = 2; a_1$ **$\frac{1}{4}$**
4. $S_n = 118.125, a_n = -5.625, r = -\frac{1}{2}; a_1$ **180**
5. $S_n = 183, r = -3, n = 5; a_1$ **3**
6. $S_n = 1705, r = 4, n = 5; a_1$ **5**
7. $S_n = 52{,}084, r = -5, n = 7; a_1$ **4**
8. $S_n = 43{,}690, r = \frac{1}{4}, n = 8; a_1$ **32, 768**
9. $S_n = 381, r = 2, n = 7; a_4$ **24**

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11-4 Skills Practice

Geometric Series

Find S_n for each geometric series described.

1. $a_1 = 2, a_5 = 162, r = 3$ **242**
2. $a_1 = 4, a_6 = 12{,}500, r = 5$ **15,624**
3. $a_1 = 1, a_8 = -1, r = -1$ **0**
4. $a_1 = 4, a_n = 256, r = -2$ **172**
5. $a_1 = 1, a_n = 729, r = -3$ **547**
6. $a_1 = 2, r = -4, n = 5$ **410**
7. $a_1 = -8, r = 2, n = 4$ **−120**
8. $a_1 = 3, r = -2, n = 12$ **−4095**
9. $a_1 = 8, r = 3, n = 5$ **968**
10. $a_1 = 6, a_n = \frac{3}{8}, r = \frac{1}{2}$ **$\frac{93}{8}$**
11. $a_1 = 8, r = \frac{1}{2}, n = 7$ **$\frac{127}{8}$**
12. $a_1 = 2, r = -\frac{1}{2}, n = 6$ **$\frac{21}{16}$**

Find the sum of each geometric series.

13. $4 + 8 + 16 + \ldots$ to 5 terms **124**
14. $-1 - 3 - 9 - \ldots$ to 6 terms **−364**
15. $3 + 6 + 12 + \ldots$ to 5 terms **93**
16. $-15 + 30 - 60 + \ldots$ to 7 terms **−645**
17. $\sum_{n=1}^{4} 3^{n-1}$ **40**
18. $\sum_{n=1}^{5} (-2)^{n-1}$ **11**
19. $\sum_{n=1}^{4} \left(\frac{1}{3}\right)^{n-1}$ **$\frac{40}{27}$**
20. $\sum_{n=1}^{9} 2(-3)^{n-1}$ **9842**

Find the indicated term for each geometric series described.

21. $S_n = 1275, a_n = 640, r = 2; a_1$ **5**
22. $S_n = -40, a_n = -54, r = -3; a_1$ **2**
23. $S_n = 99, n = 5, r = -\frac{1}{2}; a_1$ **144**
24. $S_n = 39{,}360, n = 8, r = 3; a_1$ **12**

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11-4 Practice

Geometric Series

Find S_n for each geometric series described.

1. $a_1 = 2, a_6 = 64, r = 2$ **126**
2. $a_1 = 160, a_5 = 5, r = \frac{1}{2}$ **305**
3. $a_1 = -3, a_n = -192, r = -2$ **−129**
4. $a_1 = -81, a_n = -16, r = -\frac{2}{3}$ **−55**
5. $a_1 = -3, a_n = 3072, r = -4$ **2457**
6. $a_1 = 54, a_6 = \frac{2}{9}, r = \frac{1}{3}$ **$\frac{728}{9}$**
7. $a_1 = 5, r = 3, n = 9$ **49,205**
8. $a_1 = -6, r = -1, n = 21$ **−6**
9. $a_1 = -6, r = -3, n = 7$ **−3282**
10. $a_1 = -9, r = \frac{2}{3}, n = 4$ **$-\frac{65}{3}$**
11. $a_1 = \frac{1}{3}, r = 3, n = 10$ **$\frac{29{,}524}{3}$**
12. $a_1 = 16, r = -1.5, n = 6$ **−66.5**

Find the sum of each geometric series.

13. $162 + 54 + 18 + \ldots$ to 6 terms **$\frac{728}{3}$**
14. $2 + 4 + 8 + \ldots$ to 8 terms **510**
15. $64 - 96 + 144 - \ldots$ to 7 terms **463**
16. $\frac{1}{9} - \frac{1}{3} + 1 - \ldots$ to 6 terms **$-\frac{182}{9}$**
17. $\sum_{n=1}^{8} (-3)^{n-1}$ **−1640**
18. $\sum_{n=1}^{9} 5(-2)^{n-1}$ **855**
19. $\sum_{n=1}^{5} -1(4)^{n-1}$ **−341**
20. $\sum_{n=1}^{6} \left(\frac{1}{2}\right)^{n-1}$ **$\frac{63}{32}$**
21. $\sum_{n=1}^{10} 2560\left(\frac{1}{2}\right)^{n-1}$ **5115**
22. $\sum_{n=1}^{4} 9\left(\frac{2}{3}\right)^{n-1}$ **$\frac{65}{3}$**

Find the indicated term for each geometric series described.

23. $S_n = 1023, a_n = 768, r = 4; a_1$ **3**
24. $S_n = 10{,}160, a_n = 5120, r = 2; a_1$ **80**
25. $S_n = -1365, n = 12, r = -2; a_1$ **1**
26. $S_n = 665, n = 6, r = 1.5; a_1$ **32**

27. **CONSTRUCTION** A pile driver drives a post 27 inches into the ground on its first hit. Each additional hit drives the post $\frac{2}{3}$ the distance of the prior hit. Find the total distance the post has been driven after 5 hits. **$70\frac{1}{3}$ in.**

28. **COMMUNICATIONS** Hugh Moore e-mails a joke to 5 friends on Sunday morning. Each of these friends e-mails the joke to 5 of her or his friends on Monday morning, and so on. Assuming no duplication, how many people will have heard the joke by the end of Saturday, not including Hugh? **97,655 people**

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11-4 Word Problem Practice

Geometric Series

1. **BASE 10** When the common ratio of a geometric series is 10, the sum is sometimes easier to compute because we use a decimal number system. For example, what is the sum of $1 + 10 + 10^2 + 10^3 + 10^4 + 10^5$?
   111,111

2. **INVITATIONS** Amanda wants to host a party. She invites 3 friends and tells each of them to invite 3 of their friends. The 3 friends do invite 3 others and ask each of them to invite 3 more people. This invitation process goes on for 5 generations of invitations. Including herself, how many people can Amanda expect at her party?
   364

3. **TRAINING** Arnold lifts weights. He does three bench press workouts each week. For each workout, he lifts a weight 12 times. The first week he starts with 50 pounds. Each week he increases the amount that he lifts by 10%. After 10 weeks, what is the total amount of weight that Arnold has lifted during his bench press workouts? Round your answer to the nearest pound.
   28,687 lb

4. **TEACHING** A teacher teaches 8 students how to fold an origami model. Each of these students goes on to teach 8 students of their own how to fold the same model. If this teaching process goes on for n generations, how many people will know how to fold the origami model?

Generation	1	2	3	4	5	n
Number of People Taught	1	8	64	512	4096	?

   $\mathbf{\frac{8n - 1}{7}}$

CAREERS For Exercises 5-7, use the following information.

Mary begins her new career as a professor. She begins with a salary of \$50,000. Every year, her salary increases by 7%.

5. What is Mary's salary for her nth year?
   $\mathbf{50000(1.07)^{n-1}}$

6. Use sigma notation to give an expression for the total income she will receive from the university after N years.
   $\mathbf{50000 \sum_{n=1}^{N} 1.07^{n-1}}$

7. What will be her total income from the university after 20 years?
   \$2,049,774.62

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11-4 Enrichment

Annuities

An annuity is a fixed amount of money payable at given intervals. For example, suppose you wanted to set up a trust fund so that \$30,000 could be withdrawn each year for 14 years before the money ran out. Assume the money can be invested at 9%.

You must find the amount of money that needs to be invested. Call this amount A. After the third payment, the amount left is

$1.09[1.09A - 30{,}000(1 + 1.09)] - 30{,}000 = 1.09^2A - 30{,}000(1 + 1.09 + 1.09^2)$.

The results are summarized in the table below.

Payment Number	Number of Dollars Left After Payment
1	$A - 30{,}000$
2	$1.09A - 30{,}000(1 + 1.09)$
3	$1.09^2A - 30{,}000(1 + 1.09 + 1.09^2)$

1. Use the pattern shown in the table to find the number of dollars left after the fourth payment. $\mathbf{1.09^3A - 30{,}000(1 + 1.09 + 1.09^2 + 1.09^3)}$

2. Find the amount left after the tenth payment.
   $\mathbf{1.09^9A - 30{,}000(1 + 1.09 + 1.09^2 + 1.09^3 + \ldots + 1.09^9)}$

The amount left after the 14th payment is $1.09^{13}A - 30{,}000(1 + 1.09 + 1.09^2 + \ldots + 1.09^{13})$. However, there should be no money left after the 14th and final payment.

$1.09^{13}A - 30{,}000(1 + 1.09 + 1.09^2 + \ldots + 1.09^{13}) = 0$

Notice that $1 + 1.09 + 1.09^2 + \ldots + 1.09^{13}$ is a geometric series where $a_1 = 1$, $a_n = 1.09^{13}$, $n = 14$ and $r = 1.09$.

Using the formula for S_n,

$1 + 1.09 + 1.09^2 + \ldots + 1.09^{13} = \frac{a_1 - a_1r^n}{1 - r} = \frac{1 - 1.09^{14}}{1 - 1.09} = \frac{1 - 1.09^{14}}{-0.09}$.

3. Show that when you solve for A you get $A = \frac{30{,}000}{0.09}\left(\frac{1.09^{14} - 1}{1.09^{13}}\right)$.
   $\mathbf{1.09^{13}A - 30{,}000\left(\frac{1 - 1.09^{14}}{-0.09}\right) = 0}$ **results in stated expression for *A*.**

Therefore, to provide \$30,000 for 14 years where the annual interest rate is 9%, you need $\frac{30{,}000}{0.09}\left(\frac{1.09^{14} - 1}{1.09^{13}}\right)$ dollars.

4. Use a calculator to find the value of A in problem 3. **\$254,607**

In general, if you wish to provide P dollars for each of n years at an annual rate of r%, you need A dollars where

$\left(1 + \frac{r}{100}\right)^{n-1}A - P\left[1 + \left(1 + \frac{r}{100}\right) + \left(1 + \frac{r}{100}\right)^2 + \ldots + \left(1 + \frac{r}{100}\right)^{n-1}\right] = 0.$

You can solve this equation for A, given P, n, and r.

Lesson 11-4

Answers

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11-5 Lesson Reading Guide

Infinite Geometric Series

Get Ready for the Lesson

Read the introduction to Lesson 11-5 in your textbook.

Note the following powers of 0.6: $0.6^1 = 0.6$; $0.6^2 = 0.36$; $0.6^3 = 0.216$; $0.6^4 = 0.1296$; $0.6^5 = 0.07776$; $0.6^6 = 0.046656$; $0.6^7 = 0.0279936$. If a ball is dropped from a height of 10 feet and bounces back to 60% of its previous height on each bounce, after how many bounces will it bounce back to a height of less than 1 foot? **5 bounces**

Read the Lesson

1. Consider the formula $S = \frac{a_1}{1 - r}$.
   - **a.** What is the formula used to find? **the sum of an infinite geometric series**
   - **b.** What do each of the following represent?
     - S: **the sum**
     - a_1: **the first term**
     - r: **the common ratio**
   - **c.** For what values of r does an infinite geometric sequence have a sum? **$-1 < r < 1$**
   - **d.** Rewrite your answer for part d as an absolute value inequality. **$|r| < 1$**

2. For each of the following geometric series, give the values of a_1 and r. Then state whether the sum of the series exists. (Do not actually find the sum.)
   - **a.** $\frac{2}{3} + \frac{2}{9} + \frac{2}{27} + \ldots$ $a_1 =$ **$\frac{2}{3}$** $r =$ **$\frac{1}{3}$** Does the sum exist? **yes**
   - **b.** $2 - 1 + \frac{1}{2} - \frac{1}{4} + \ldots$ $a_1 =$ **2** $r =$ **$-\frac{1}{2}$** Does the sum exist? **yes**
   - **c.** $\sum_{i=1}^{\infty} 3^i$ $a_1 =$ **3** $r =$ **3** Does the sum exist? **no**

Remember What You Learned

3. One good way to remember something is to relate it to something you already know. How can you use the formula $S_n = \frac{a_1(1 - r^n)}{1 - r}$ that you learned in Lesson 11-4 for finding the sum of a geometric series to help you remember the formula for finding the sum of an infinite geometric series? **Sample answer: If $-1 < r < 1$, then as n gets large, r^n approaches 0, so $1 - r^n$ approaches 1. Therefore, S_n approaches $\frac{a_1 \cdot 1}{1 - r}$, or $\frac{a_1}{1 - r}$.**

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11-5 Study Guide and Intervention

Infinite Geometric Series

Infinite Geometric Series A geometric series that does not end is called an **infinite geometric series**. Some infinite geometric series have sums, but others do not because the **partial sums** increase without approaching a limiting value.

Sum of an Infinite Geometric Series	$S = \frac{a_1}{1 - r}$ for $-1 < r < 1$. If $|r| \geq 1$, the infinite geometric series does not have a sum.

Example **Find the sum of each infinite geometric series, if it exists.**

a. 75 + 15 + 3 + ...

First, find the value of r to determine if the sum exists. $a_1 = 75$ and $a_2 = 15$, so $r = \frac{15}{75}$ or $\frac{1}{5}$. Since $\left|\frac{1}{5}\right| < 1$, the sum exists. Now use the formula for the sum of an infinite geometric series.

$S = \frac{a_1}{1 - r}$ Sum formula

$= \frac{75}{1 - \frac{1}{5}}$ $a_1 = 75, r = \frac{1}{5}$

$= \frac{75}{\frac{4}{5}}$ or 93.75 Simplify.

The sum of the series is 93.75.

b. $\sum_{n=1}^{\infty} 48\left(-\frac{1}{3}\right)^{n-1}$

In this infinite geometric series, $a_1 = 48$ and $r = -\frac{1}{3}$.

$S = \frac{a_1}{1 - r}$ Sum formula

$= \frac{48}{1 - \left(-\frac{1}{3}\right)}$ $a_1 = 48, r = -\frac{1}{3}$

$= \frac{48}{\frac{4}{3}}$ or 36 Simplify.

Thus $\sum_{n=1}^{\infty} 48\left(-\frac{1}{3}\right)^{n-1} = 36$.

Exercises

Find the sum of each infinite geometric series, if it exists.

1. $a_1 = -7, r = \frac{5}{8}$ **$-18\frac{2}{3}$**
2. $1 + \frac{5}{4} + \frac{25}{16} + \ldots$ **does not exist**
3. $a_1 = 4, r = \frac{1}{2}$ **8**
4. $\frac{2}{9} + \frac{5}{27} + \frac{25}{162} + \ldots$ **$1\frac{1}{3}$**
5. $15 + 10 + 6\frac{2}{3} + \ldots$ **45**
6. $18 - 9 + 4\frac{1}{2} - 2\frac{1}{4} + \ldots$ **12**
7. $\frac{1}{10} + \frac{1}{20} + \frac{1}{40} + \ldots$ **$\frac{1}{5}$**
8. $1000 + 800 + 640 + \ldots$ **5000**
9. $6 - 12 + 24 - 48 + \ldots$ **does not exist**
10. $\sum_{n=1}^{\infty} 50\left(\frac{4}{5}\right)^{n-1}$ **250**
11. $\sum_{k=1}^{\infty} 22\left(-\frac{1}{2}\right)^{k-1}$ **$14\frac{2}{3}$**
12. $\sum_{s=1}^{\infty} 24\left(\frac{7}{12}\right)^{s-1}$ **$57\frac{3}{5}$**

Lesson 11-5

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11-5 Study Guide and Intervention *(continued)*

Infinite Geometric Series

Repeating Decimals A repeating decimal represents a fraction. To find the fraction, write the decimal as an infinite geometric series and use the formula for the sum.

Example **Write each repeating decimal as a fraction.**

a. $0.\overline{42}$

Write the repeating decimal as a sum.

$0.\overline{42} = 0.42424242...$

$= \frac{42}{100} + \frac{42}{10{,}000} + \frac{42}{1{,}000{,}000} + ...$

In this series $a_1 = \frac{42}{100}$ and $r = \frac{1}{100}$.

$S = \frac{a_1}{1-r}$ — Sum formula

$= \frac{\frac{42}{100}}{1 - \frac{1}{100}}$ — $a_1 = \frac{42}{100}$, $r = \frac{1}{100}$

$= \frac{\frac{42}{100}}{\frac{99}{100}}$ — Subtract.

$= \frac{42}{99}$ or $\frac{14}{33}$ — Simplify.

Thus $0.\overline{42} = \frac{14}{33}$.

b. $0.5\overline{24}$

Let $S = 0.5\overline{24}$.

$S = 0.5242424...$ — Write as a repeating decimal.

$1000S = 524.242424...$ — Multiply each side by 1000.

$10S = 5.242424...$ — Mulitply each side by 10.

$990S = 519$ — Subtract the third equation from the second equation.

$S = \frac{519}{990}$ or $\frac{173}{330}$ — Simplify.

Thus, $0.5\overline{24} = \frac{173}{330}$

Exercises

Write each repeating decimal as a fraction.

1. $0.\overline{2}$ $\frac{2}{9}$	**2.** $0.\overline{8}$ $\frac{8}{9}$	**3.** $0.\overline{30}$ $\frac{10}{33}$	**4.** $0.\overline{87}$ $\frac{29}{33}$
5. $0.\overline{10}$ $\frac{10}{99}$	**6.** $0.\overline{54}$ $\frac{6}{11}$	**7.** $0.\overline{75}$ $\frac{25}{33}$	**8.** $0.\overline{18}$ $\frac{2}{11}$
9. $0.\overline{62}$ $\frac{62}{99}$	**10.** $0.\overline{72}$ $\frac{8}{11}$	**11.** $0.0\overline{72}$ $\frac{4}{55}$	**12.** $0.0\overline{45}$ $\frac{1}{22}$
13. $0.0\overline{6}$ $\frac{1}{15}$	**14.** $0.0\overline{138}$ $\frac{23}{1665}$	**15.** $0.\overline{0138}$ $\frac{46}{3333}$	**16.** $0.0\overline{81}$ $\frac{9}{110}$
17. $0.2\overline{45}$ $\frac{27}{110}$	**18.** $0.4\overline{36}$ $\frac{24}{55}$	**19.** $0.5\overline{4}$ $\frac{49}{90}$	**20.** $0.8\overline{63}$ $\frac{19}{22}$

Chapter 11 — 36 — *Glencoe Algebra 2*

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11-5 Skills Practice

Infinite Geometric Series

Find the sum of each infinite geometric series, if it exists.

1. $a_1 = 1, r = \frac{1}{2}$ **2**	**2.** $a_1 = 5, r = -\frac{2}{5}$ $\frac{25}{7}$
3. $a_1 = 8, r = 2$ **does not exist**	**4.** $a_1 = 6, r = \frac{1}{2}$ **12**
5. $4 + 2 + 1 + \frac{1}{2} + ...$ **8**	**6.** $540 - 180 + 60 - 20 + ...$ **405**
7. $5 + 10 + 20 + ...$ **does not exist**	**8.** $-336 + 84 - 21 + ...$ **−268.8**
9. $125 + 25 + 5 + ...$ **156.25**	**10.** $9 - 1 + \frac{1}{9} - ...$ $\frac{81}{10}$
11. $\frac{3}{4} + \frac{9}{4} + \frac{27}{4} + ...$ **does not exist**	**12.** $\frac{1}{3} + \frac{1}{9} + \frac{1}{27} + ...$ $\frac{1}{2}$
13. $5 + 2 + 0.8 + ...$ $\frac{25}{3}$	**14.** $9 + 6 + 4 + ...$ **27**
15. $\sum_{n=1}^{\infty} 10\left(\frac{1}{2}\right)^{n-1}$ **20**	**16.** $\sum_{n=1}^{\infty} 6\left(-\frac{1}{3}\right)^{n-1}$ $\frac{9}{2}$
17. $\sum_{n=1}^{\infty} 15\left(\frac{2}{5}\right)^{n-1}$ **25**	**18.** $\sum_{n=1}^{\infty} \left(-\frac{4}{3}\right)\left(\frac{1}{3}\right)^{n-1}$ **−2**

Write each repeating decimal as a fraction.

19. $0.\overline{4}$ $\frac{4}{9}$	**20.** $0.\overline{8}$ $\frac{8}{9}$
21. $0.\overline{27}$ $\frac{3}{11}$	**22.** $0.\overline{67}$ $\frac{67}{99}$
23. $0.\overline{54}$ $\frac{6}{11}$	**24.** $0.\overline{375}$ $\frac{125}{333}$
25. $0.\overline{641}$ $\frac{641}{999}$	**26.** $0.\overline{171}$ $\frac{19}{111}$

Chapter 11 — 37 — *Glencoe Algebra 2*

Lesson 11-5

Answers

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11-5 Practice

Infinite Geometric Series

Find the sum of each infinite geometric series, if it exists.

1. $a_1 = 35, r = \frac{2}{7}$ **49**
2. $a_1 = 26, r = \frac{1}{2}$ **52**
3. $a_1 = 98, r = -\frac{3}{4}$ **56**
4. $a_1 = 42, r = \frac{6}{5}$ **does not exist**
5. $a_1 = 112, r = -\frac{3}{5}$ **70**
6. $a_1 = 500, r = \frac{1}{5}$ **625**
7. $a_1 = 135, r = -\frac{1}{2}$ **90**
8. $18 - 6 + 2 - \ldots$ $\frac{27}{2}$
9. $2 + 6 + 18 + \ldots$ **does not exist**
10. $6 + 4 + \frac{8}{3} + \ldots$ **18**
11. $\frac{4}{25} + \frac{2}{5} + 1 + \ldots$ **does not exist**
12. $10 + 1 + 0.1 + \ldots$ $\frac{100}{9}$
13. $100 + 20 + 4 + \ldots$ **125**
14. $-270 + 135 - 67.5 + \ldots$ **−180**
15. $0.5 + 0.25 + 0.125 + \ldots$ **1**
16. $\frac{7}{10} + \frac{7}{100} + \frac{7}{1000} + \ldots$ $\frac{7}{9}$
17. $0.8 + 0.08 + 0.008 + \ldots$ $\frac{8}{9}$
18. $\frac{1}{12} - \frac{1}{6} + \frac{1}{3} - \ldots$ **does not exist**
19. $3 + \frac{9}{7} + \frac{27}{49} + \ldots$ $\frac{21}{4}$
20. $0.3 - 0.003 + 0.00003 - \ldots$ $\frac{30}{101}$
21. $0.06 + 0.006 + 0.0006 + \ldots$ $\frac{1}{15}$
22. $\frac{2}{3} - 2 + 6 - \ldots$ **does not exist**
23. $\sum_{n=1}^{\infty} 3\left(\frac{1}{4}\right)^{n-1}$ **4**
24. $\sum_{n=1}^{\infty} \frac{2}{3}\left(-\frac{3}{4}\right)^{n-1}$ $\frac{8}{21}$
25. $\sum_{n=1}^{\infty} 18\left(\frac{2}{3}\right)^{n-1}$ **54**
26. $\sum_{n=1}^{\infty} 5(-0.1)^{n-1}$ $\frac{50}{11}$

Write each repeating decimal as a fraction.

27. $0.\overline{6}$ $\frac{2}{3}$
28. $0.\overline{09}$ $\frac{1}{11}$
29. $0.\overline{43}$ $\frac{43}{99}$
30. $0.\overline{27}$ $\frac{3}{11}$
31. $0.\overline{243}$ $\frac{9}{37}$
32. $0.\overline{84}$ $\frac{28}{33}$
33. $0.\overline{990}$ $\frac{110}{111}$
34. $0.\overline{150}$ $\frac{50}{333}$

35. **PENDULUMS** On its first swing, a pendulum travels 8 feet. On each successive swing, the pendulum travels $\frac{4}{5}$ the distance of its previous swing. What is the total distance traveled by the pendulum when it stops swinging? **40 ft**

36. **ELASTICITY** A ball dropped from a height of 10 feet bounces back $\frac{9}{10}$ of that distance. With each successive bounce, the ball continues to reach $\frac{9}{10}$ of its previous height. What is the total vertical distance (both up and down) traveled by the ball when it stops bouncing? (*Hint*: Add the total distance the ball falls to the total distance it rises.) **190 ft**

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11-5 Word Problem Practice

Infinite Geometric Series

1. **PARADOX** If the formula for the sum of a geometric series is applied to the series whose first term is 1 and common ratio is 2, the result is the equation $-1 = 1 + 2 + 2^2 + 2^3 + \ldots$. Is this equality really true? Explain.
No, it is not true. An infinite geometric series must have $r < 1$ to have a sum.

2. **BOUNDS** Can the sum of an infinite geometric series whose first term is 1 be as large as we wish?
Yes, it can. The sum is $\frac{1}{1-r}$, where r is the common ratio. This expression increases without bound as r approaches 1 from below.

3. **BASES** The infinite repeating decimal 0.999… is equal to 1. This can be shown by using the sum of a geometric series with common ratio $\frac{1}{10}$ and first term $\frac{9}{10}$. In a similar vein, compute the sum of the infinite geometric series $\frac{b-1}{b} + \frac{b-1}{b^2} + \frac{b-1}{b^3} + \cdots$, where b is a positive integer greater than 1. How is this sum related to the fact that 0.999… = 1?
1; It is similar to 0.999… in base b.

4. **CLIMBING** A robot is designed to climb a wall each time a button is pressed. The first time the button is pressed, it climbs 10 feet. Each time after, the robot climbs only 75% of what it climbed the last time. What is the smallest upper limit on how high the robot can climb?
40 ft

INSTALLMENTS For Exercises 5-7, use the following information.

Jade lends Jack a 100-pound chunk of pure gold for one year. After one year, she wants to start getting the gold back. One year later, Jack begins returning the gold, by giving Jade 1 pound of gold. The next day, Jack gives her 0.99 pounds of gold. The next day, Jack gives her $(0.99)^2$ or 0.9801 pounds of gold. Each successive day, Jack gives 0.99 times as much gold as the previous day.

5. How much gold does Jade get back on the nth day that Jack begins returning the gold.
$(0.99)^{n-1}$ lb

6. How much gold has Jade received after 10 days? 100 days? Infinitely many days? Round your answers to the nearest hundredth of a pound.
9.56 lb after 10 days; 63.40 lb after 100 days; 100 lb after infinitely many days

7. Will Jade have all her gold back at any specific date in the future? Explain.
No. At the rate Jack is returning the gold, there will always be a small amount of the gold that will never be returned.

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11-5 Enrichment

Infinite Continued Fractions

Some infinite expressions are actually equal to real numbers! The infinite continued fraction at the right is one example.

$$x = 1 + \cfrac{1}{1 + \cfrac{1}{1 + \cfrac{1}{1 + \cfrac{1}{1 + \dots}}}}$$

If you use x to stand for the infinite fraction, then the entire denominator of the first fraction on the right is also equal to x. This observation leads to the following equation:

$$x = 1 + \frac{1}{x}$$

Write a decimal for each continued fraction.

1. $1 + \frac{1}{1}$ **2**
2. $1 + \cfrac{1}{1 + \frac{1}{1}}$ **1.5**
3. $1 + \cfrac{1}{1 + \cfrac{1}{1 + \frac{1}{1}}}$ **$1.66\overline{6}$**
4. $1 + \cfrac{1}{1 + \cfrac{1}{1 + \cfrac{1}{1 + \frac{1}{1}}}}$ **1.6**
5. $1 + \cfrac{1}{1 + \cfrac{1}{1 + \cfrac{1}{1 + \cfrac{1}{1 + \frac{1}{1}}}}}$ **1.625**

6. The more terms you add to the fractions above, the closer their value approaches the value of the infinite continued fraction. What value do the fractions seem to be approaching? **about 1.6**

7. Rewrite $x = 1 + \frac{1}{x}$ as a quadratic equation and solve for x.
$x^2 - x - 1 = 0$; $x = \frac{1 \pm \sqrt{5}}{2}$; $x \approx 1.618$ or -0.618
(The positive root is the value of the infinite fraction, because the original fraction is clearly not negative.)

8. Find the value of the following infinite continued fraction.
$$3 + \cfrac{1}{3 + \cfrac{1}{3 + \cfrac{1}{3 + \cfrac{1}{3 + \dots}}}}$$
$x = 3 + \frac{1}{x}$; $x = \frac{3 + \sqrt{13}}{2}$ or about 3.30

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11-6 Lesson Reading Guide

Recursion and Special Sequences

Get Ready for the Lesson

Read the introduction to Lesson 11-6 in your textbook.

What are the next three numbers in the sequence that gives the number of shoots corresponding to each month? **8, 13, 21**

Read the Lesson

1. Consider the sequence in which $a_1 = 4$ and $a_n = 2a_{n-1} + 5$.
   a. Explain why this is a *recursive* formula. **Sample answer: Each term is found from the value of the previous term.**
   b. Explain in your own words how to find the first four terms of this sequence. (Do not actually find any terms after the first.) **Sample answer: The first term is 4. To find the second term, double the first term and add 5. To find the third term, double the second term and add 5. To find the fourth term, double the third term and add 5.**
   c. What happens to the terms of this sequence as n increases? **Sample answer: They keep getting larger and larger.**

2. Consider the function $f(x) = 3x - 1$ with an initial value of $x_0 = 2$.
   a. What does it mean to *iterate* this function?
   to compose the function with itself repeatedly
   b. Fill in the blanks to find the first three iterates. The blanks that follow the letter x are for subscripts.

   $x_1 = f(x_{\underline{0}}) = f(\underline{2}) = 3(\underline{2}) - 1 = \underline{6} - 1 = \underline{5}$

   $x_2 = f(x_{\underline{1}}) = f(\underline{5}) = 3(\underline{5}) - 1 = \underline{14}$

   $x_3 = f(x_{\underline{2}}) = f(\underline{14}) = 3(\underline{14}) - 1 = \underline{41}$

   c. As this process continues, what happens to the values of the iterates?
   Sample answer: They keep getting larger and larger.

Remember What You Learned

3. Use a dictionary to find the meanings of the words *recurrent* and *iterate*. How can the meanings of these words help you to remember the meaning of the mathematical terms *recursive* and *iteration*? How are these ideas related? **Sample answer: *Recurrent* means happening repeatedly, while *iterate* means to repeat a process or operation. A *recursive* formula is used repeatedly to find the value of one term of a sequence based on the previous term. *Iteration* means to compose a function with it self repeatedly. Both ideas have to do with repetition—doing the same thing over and over again.**

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11-6 Study Guide and Intervention

Recursion and Special Sequences

Special Sequences In a **recursive formula**, each succeeding term is formulated from one or more previous terms. A recursive formula for a sequence has two parts:

1. the value(s) of the first term(s), and
2. an equation that shows how to find each term from the term(s) before it.

Example **Find the first five terms of the sequence in which $a_1 = 6$, $a_2 = 10$, and $a_n = 2a_{n-2}$ for $n \geq 3$.**

$a_1 = 6$
$a_2 = 10$
$a_3 = 2a_1 = 2(6) = 12$
$a_4 = 2a_2 = 2(10) = 20$
$a_5 = 2a_3 = 2(12) = 24$

The first five terms of the sequence are 6, 10, 12, 20, 24.

Exercises

Find the first five terms of each sequence.

1. $a_1 = 1, a_2 = 1, a_n = 2(a_{n-1} + a_{n-2}), n \geq 3$ **1, 1, 4, 10, 28**

2. $a_1 = 1, a_n = \frac{1}{1 + a_{n-1}}, n \geq 2$ **$1, \frac{1}{2}, \frac{2}{3}, \frac{3}{5}, \frac{5}{8}$**

3. $a_1 = 3, a_n = a_{n-1} + 2(n - 2), n \geq 2$ **3, 3, 5, 9, 15**

4. $a_1 = 5, a_n = a_{n-1} + 2, n \geq 2$ **5, 7, 9, 11, 13**

5. $a_1 = 1, a_n = (n - 1)a_{n-1}, n \geq 2$ **1, 1, 2, 6, 24**

6. $a_1 = 7, a_n = 4a_{n-1} - 1, n \geq 2$ **7, 27, 107, 427, 1707**

7. $a_1 = 3, a_2 = 4, a_n = 2a_{n-2} + 3a_{n-1}, n \geq 3$ **3, 4, 18, 62, 222**

8. $a_1 = 0.5, a_n = a_{n-1} + 2n, n \geq 2$ **0.5, 4.5, 10.5, 18.5, 28.5**

9. $a_1 = 8, a_2 = 10, a_n = \frac{a_{n-2}}{a_{n-1}}, n \geq 3$ **8, 10, 0.8, 12.5, 0.064**

10. $a_1 = 100, a_n = \frac{a_{n-1}}{n}, n \geq 2$ **$100, 50, \frac{50}{3}, \frac{25}{6}, \frac{5}{6}$**

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11-6 Study Guide and Intervention *(continued)*

Recursion and Special Sequences

Iteration Combining composition of functions with the concept of recursion leads to the process of **iteration**. Iteration is the process of composing a function with itself repeatedly.

Example **Find the first three iterates of $f(x) = 4x - 5$ for an initial value of $x_0 = 2$.**

To find the first iterate, find the value of the function for $x_0 = 2$

$x_1 = f(x_0)$ Iterate the function.
$= f(2)$ $x_0 = 2$
$= 4(2) - 5$ or 3 Simplify.

To find the second iteration, find the value of the function for $x_1 = 3$.

$x_2 = f(x_1)$ Iterate the function.
$= f(3)$ $x_1 = 3$
$= 4(3) - 5$ or 7 Simplify.

To find the third iteration, find the value of the function for $x_2 = 7$.

$x_3 = f(x_2)$ Iterate the function.
$= f(7)$ $x_2 = 7$
$= 4(7) - 5$ or 23 Simplify.

The first three iterates are 3, 7, and 23.

Exercises

Find the first three iterates of each function for the given initial value.

1. $f(x) = x - 1; x_0 = 4$ **3, 2, 1**

2. $f(x) = x^2 - 3x; x_0 = 1$ **−2, 10, 70**

3. $f(x) = x^2 + 2x + 1; x_0 = -2$ **1, 4, 25**

4. $f(x) = 4x - 6; x_0 = -5$ **−26, −110, −446**

5. $f(x) = 6x - 2; x_0 = 3$ **16, 94, 562**

6. $f(x) = 100 - 4x; x_0 = -5$ **120, −380, 1620**

7. $f(x) = 3x - 1; x_0 = 47$ **140, 419, 1256**

8. $f(x) = x^3 - 5x^2; x_0 = 1$ **−4, −144, −3,089,664**

9. $f(x) = 10x - 25; x_0 = 2$ **−5, −75, −775**

10. $f(x) = 4x^2 - 9; x_0 = -1$ **−5, 91, 33,115**

11. $f(x) = 2x^2 + 5; x_0 = -4$ **37, 2743, 15,048,103**

12. $f(x) = \frac{x - 1}{x + 2}; x_0 = 1$ **$0, -\frac{1}{2}, -1$**

13. $f(x) = \frac{1}{2}(x + 11); x_0 = 3$ **7, 9, 10**

14. $f(x) = \frac{3}{x}; x_0 = 9$ **$\frac{1}{3}, 9, \frac{1}{3}$**

15. $f(x) = x - 4x^2; x_0 = 1$ **−3, −39, −6123**

16. $f(x) = x + \frac{1}{x}; x_0 = 2$ **2.5, 2.9, about 3.245**

17. $f(x) = x^3 - 5x^2 + 8x - 10; x_0 = 1$ **−6, −454, −94,610,886**

18. $f(x) = x^3 - x^2; x_0 = -2$ **−12, −1872, −6,563,711,232**

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11-6 Skills Practice

Recursion and Special Sequences

Find the first five terms of each sequence.

1. $a_1 = 4, a_{n+1} = a_n + 7$
4, 11, 18, 25, 32

2. $a_1 = -2, a_{n+1} = a_n + 3$
−2, 1, 4, 7, 10

3. $a_1 = 5, a_{n+1} = 2a_n$
5, 10, 20, 40, 80

4. $a_1 = -4, a_{n+1} = 6 - a_n$
−4, 10, −4, 10, −4

5. $a_1 = 1, a_{n+1} = a_n + n$
1, 2, 4, 7, 11

6. $a_1 = -1, a_{n+1} = n - a_n$
−1, 2, 0, 3, 1

7. $a_1 = -6, a_{n+1} = a_n + n + 1$
−6, −4, −1, 3, 8

8. $a_1 = 8, a_{n+1} = a_n - n - 2$
8, 5, 1, −4, −10

9. $a_1 = -3, a_{n+1} = 2a_n + 7$
−3, 1, 9, 25, 57

10. $a_1 = 4, a_{n+1} = -2a_n - 5$
4, −13, 21, −47, 89

11. $a_1 = 0, a_2 = 1, a_{n+1} = a_n + a_{n-1}$
0, 1, 1, 2, 3

12. $a_1 = -1, a_2 = -1, a_{n+1} = a_n - a_{n-1}$
−1, −1, 0, 1, 1

13. $a_1 = 3, a_2 = -5, a_{n+1} = -4a_n + a_{n-1}$
3, −5, 23, −97, 411

14. $a_1 = -3, a_2 = 2, a_{n+1} = a_{n-1} - a_n$
−3, 2, −5, 7, −12

Find the first three iterates of each function for the given initial value.

15. $f(x) = 2x - 1, x_0 = 3$ **5, 9, 17**

16. $f(x) = 5x - 3, x_0 = 2$ **7, 32, 157**

17. $f(x) = 3x + 4, x_0 = -1$ **1, 7, 25**

18. $f(x) = 4x + 7, x_0 = -5$ **−13, −45, −173**

19. $f(x) = -x - 3, x_0 = 10$ **−13, 10, −13**

20. $f(x) = -3x + 6, x_0 = 6$ **−12, 42, −120**

21. $f(x) = -3x + 4, x_0 = 2$ **−2, 10, −26**

22. $f(x) = 6x - 5, x_0 = 1$ **1, 1, 1**

23. $f(x) = 7x + 1, x_0 = -4$
−27, −188, −1315

24. $f(x) = x^2 - 3x, x_0 = 5$
10, 70, 4690

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11-6 Practice

Recursion and Special Sequences

Find the first five terms of each sequence.

1. $a_1 = 3, a_{n+1} = a_n + 5$
3, 8, 13, 18, 23

2. $a_1 = -7, a_{n+1} = a_n + 8$
−7, 1, 9, 17, 25

3. $a_1 = -3, a_{n+1} = 3a_n + 2$
−3, −7, −19, −55, −163

4. $a_1 = -8, a_{n+1} = 10 - a_n$
−8, 18, −8, 18, −8

5. $a_1 = 4, a_{n+1} = n - a_n$
4, −3, 5, −2, 6

6. $a_1 = -3, a_{n+1} = 3a_n$
−3, −9, −27,−81, −243

7. $a_1 = 4, a_{n+1} = -3a_n + 4$
4, −8, 28, −80, 244

8. $a_1 = 2, a_{n+1} = -4a_n - 5$
2, −13, 47, −193, 767

9. $a_1 = 3, a_2 = 1, a_{n+1} = a_n - a_{n-1}$
3, 1, −2, −3, −1

10. $a_1 = -1, a_2 = 5, a_{n+1} = 4a_{n-1} - a_n$
−1, 5, −9, 29, −65

11. $a_1 = 2, a_2 = -3, a_{n+1} = 5a_n - 8a_{n-1}$
2, −3, −31, −131, −407

12. $a_1 = -2, a_2 = 1, a_{n+1} = -2a_n + 6a_{n-1}$
−2, 1, −14, 34, −152

Find the first three iterates of each function for the given initial value.

13. $f(x) = 3x + 4, x_0 = -1$ **1, 7, 25**

14. $f(x) = 10x + 2, x_0 = -1$ **−8, −78, −778**

15. $f(x) = 8 + 3x, x_0 = 1$ **11, 41, 131**

16. $f(x) = 8 - x, x_0 = -3$ **11, −3, 11**

17. $f(x) = 4x + 5, x_0 = -1$ **1, 9, 41**

18. $f(x) = 5(x + 3), x_0 = -2$ **5, 40, 215**

19. $f(x) = -8x + 9, x_0 = 1$ **1, 1, 1**

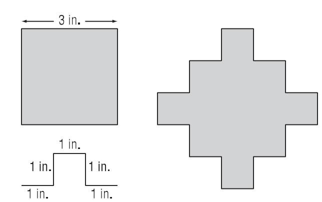
20. $f(x) = -4x^2, x_0 = -1$ **−4; −64; −16,384**

21. $f(x) = x^2 - 1, x_0 = 3$ **8, 63, 3968**

22. $f(x) = 2x^2; x_0 = 5$ **50; 5000; 50,000,000**

23. INFLATION Iterating the function $c(x) = 1.05x$ gives the future cost of an item at a constant 5% inflation rate. Find the cost of a $2000 ring in five years at 5% inflation.
$2552.56

FRACTALS For Exercises 24–27, use the following information.
Replacing each side of the square shown with the combination of segments below it gives the figure to its right.



24. What is the perimeter of the original square?
12 in.

25. What is the perimeter of the new shape? **20 in.**

26. If you repeat the process by replacing each side of the new shape by a proportional combination of 5 segments, what will the perimeter of the third shape be? **$33\frac{1}{3}$ in.**

27. What function $f(x)$ can you iterate to find the perimeter of each successive shape if you continue this process? **$f(x) = \frac{5}{3}x$**

Lesson 11-6

Answers

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11-6 Word Problem Practice

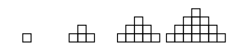
Recursion and Special Sequences

1. **GEOMETRIC SEQUENCES** The geometric sequence with first term a and common ratio r goes like this: a, ar, ar^2, ar^3, etc. It happens that this sequence can also be seen from the point of view of iterative sequences. What function $f(x)$ can be used to define the geometric sequence above iteratively?
$f(x) = rx$

2. **BACTERIA** All the bacteria in a bacterial culture divide in two every hour. Also, every hour, 1,000 bacteria are removed from the culture. If the initial population consisted of 1,100 bacteria, what are the population sizes every hour for the next four hours?
Starting with 1100, the population increases to 1200, 1400, 1800, then 2600.

3. **WORK** The company that Robert works for has a policy where the number of hours you have to work one week depends on the number of hours worked the previous week. If you worked h hours one week, then the next week you must work at least $80 - h$ hours. Robert worked 20 hours his first week with the company. From then on, he always worked the minimum number of hours required of him. Describe the number of hours Robert worked from week to week.
Robert alternated 20-hour weeks with 60-hour weeks

4. **GEOMETRY** A sequence of triangular shapes is made using squares as shown in the figure.



Let x_n be the number of squares to make the nth figure. Write a recursive formula for x_n.
$x_1 = 1$; $x_{n+1} = x_n + 2n + 1$ for $n > 0$

PATHS For Exercises 5 and 6, use the following information.

Gregory makes walking paths out of two different rectangles. One is a 1-yard by 1-yard square and the other is a 1-yard by 2-yard rectangle. He makes paths by lining up the squares and rectangles as shown in the figure.

Gregory wants to know how many different paths he can make of a fixed length. Let a_n denote the number of paths he can make of length n yards.

5. What are the first 5 values of a_n?
1, 2, 3, 5, 8

6. Write a recursive formula for a_n. Explain.
$a_1 = 1$; $a_2 = 2$; $a_{n+1} = a_n + a_{n-1}$ for $n > 1$. Every path of length $n - 1$ can be extended to a path of length $n + 1$ by adding a 1 by 2 rectangle and every path of length n can be extended to a path of length $n + 1$ by adding a 1 by 1 rectangle.

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11-6 Enrichment

Continued Fractions

The fraction below is an example of a continued fraction. Note that each fraction in the continued fraction has a numerator of 1.

$$2 + \cfrac{1}{3 + \cfrac{1}{4 + \frac{1}{5}}}$$

Example 1 **Evaluate the continued fraction above. Start at the bottom and work your way up.**

Step 1: $4 + \frac{1}{5} = \frac{20}{5} + \frac{1}{5} = \frac{21}{5}$

Step 2: $\frac{1}{\frac{21}{5}} = \frac{5}{21}$

Step 3: $3 + \frac{5}{21} = \frac{63}{21} + \frac{5}{21} = \frac{68}{21}$

Step 4: $\frac{1}{\frac{68}{21}} = \frac{21}{68}$

Step 5: $2 + \frac{21}{68} = 2\frac{21}{68}$

Example 2 **Change $\frac{25}{11}$ into a continued fraction.**

Follow the steps.

Step 1: $\frac{25}{11} = \frac{22}{11} + \frac{3}{11} = 2 + \frac{3}{11}$

Step 2: $\frac{3}{11} = \frac{1}{\frac{11}{3}}$

Step 3: $\frac{11}{3} = \frac{9}{3} + \frac{2}{3} = 3 + \frac{2}{3}$

Step 4: $\frac{2}{3} = \frac{1}{\frac{3}{2}}$

Step 5: $\frac{3}{2} = \frac{2}{2} + \frac{1}{2} = 1 + \frac{1}{2}$ Stop, because the numerator is 1.

Thus, $\frac{25}{11}$ can be written as $2 + \cfrac{1}{3 + \cfrac{1}{1 + \frac{1}{2}}}$

Evaluate each continued fraction.

1. $1 + \cfrac{1}{1 + \cfrac{1}{2 + \cfrac{1}{3 + \frac{1}{3}}}}$ **$1\frac{17}{24}$**

2. $0 + \cfrac{1}{6 + \cfrac{1}{4 + \frac{1}{2}}}$ **$\frac{9}{56}$**

3. $2 + \cfrac{1}{4 + \cfrac{1}{6 + \cfrac{1}{8 + \frac{1}{10}}}}$ **$2\frac{496}{2065}$**

4. $5 + \cfrac{1}{7 + \cfrac{1}{9 + \frac{1}{11}}}$ **$5\frac{100}{711}$**

Change each fraction into a continued fraction.

5. $\frac{75}{31}$
$2 + \cfrac{1}{2 + \cfrac{1}{2 + \cfrac{1}{1 + \cfrac{1}{1 + \frac{1}{2}}}}}$

6. $\frac{29}{8}$
$3 + \cfrac{1}{1 + \cfrac{1}{1 + \cfrac{1}{1 + \frac{1}{2}}}}$

7. $\frac{13}{19}$
$0 + \cfrac{1}{1 + \cfrac{1}{2 + \frac{1}{6}}}$

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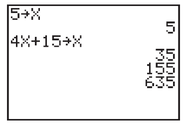
11-6 Graphing Calculator Activity

Recursion and Iteration

A graphing calculator can be used to perform iterations and recursions.

Example 1 **Find the first 3 iterates of $f(x) = 4x + 15$ if $x_0 = 5$.**

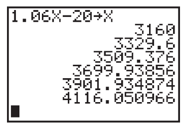
Store x_0 in X. Then enter the expression on the home screen. Store the result to X. Repeat the calculation for each iterate.
Keystrokes: 5 [STO▸] [X,T,θ,n] [ENTER] 4 [X,T,θ,n] [+] 15 [STO▸] [X,T,θ,n] [ENTER] [ENTER] [ENTER].



$x_1 = 35$, $x_2 = 155$, and $x_3 = 635$

Example 2 **A savings account has an initial balance of $3000.00. At the end of each year, the bank pays 6% interest and charges a $20 annual fee. Find the account balance after 6 years.**

Store the initial value and enter an expression to calculate the balance at the end of a year.
Keystrokes: 3000 [STO▸] [X,T,θ,n] [ENTER] 1.06 [X,T,θ,n] [—] 20 [STO▸] [X,T,θ,n] [ENTER] [ENTER] [ENTER] [ENTER] [ENTER] [ENTER].



At the end of six years, the account has a balance of $4116.05.

Exercises

Find the first three iterates of each function.

1. $f(x) = 6x + 12$ if $x_0 = 5$
$x_1 = 42, x_2 = 264, x_3 = 1596$

2. $f(x) = 2x^2 - 3$ if $x_0 = -1$
$x_1 = -1, x_2 = -1, x_3 = -1$

3. $f(x) = x^2 - 4x + 5$ if $x_0 = 1$
$x_1 = 2, x_2 = 1, x_3 = 2$

4. $f(x) = 2x^2 + 2x + 1$ if $x_0 = \frac{1}{2}$
$x_1 = \frac{5}{2}, x_2 = \frac{37}{2}, x_3 = \frac{1445}{2}$

A bank account has an initial balance of $11,250.00. Interest is paid at the end of each year. Find the account balance under the given interest rate after the stated time period.

5. 3.8%, 2 years
$12,121.25

6. 4.75%, 5 years
$14,188.05

7. 6.05%, 10 years
$20,242.27

8. 7.44%, 15 years
$33,009.77

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11-7 Lesson Reading Guide

The Binomial Theorem

Lesson 11-7

Get Ready for the Lesson

Read the introduction to Lesson 11-7 in your textbook.

- If a family has four children, list the sequences of births of girls and boys that result in three girls and one boy. **BGGG GBGG GGBG GGGB**
- Describe a way to figure out how many such sequences there are without listing them. **Sample answer: The boy could be the first, second, third, or fourth child, so there are four sequences with three girls and one boy.**

Read the Lesson

1. Consider the expansion of $(w + z)^5$.

a. How many terms does this expansion have? **6**

b. In the second term of the expansion, what is the exponent of w? **4**

What is the exponent of z? **1**

What is the coefficient of the second term? **5**

c. In the fourth term of the expansion, what is the exponent of w? **2**

What is the exponent of z? **3**

What is the coefficient of the fourth term? **10**

d. What is the last term of this expansion? z^5

2. a. State the definition of a *factorial* in your own words. (Do not use mathematical symbols in your definition.) **Sample answer: The factorial of any positive integer is the product of that integer and all the smaller integers down to one. The factorial of zero is one.**

b. Write out the product that you would use to calculate 10!. (Do not actually calculate the product.) $10 \cdot 9 \cdot 8 \cdot 7 \cdot 6 \cdot 5 \cdot 4 \cdot 3 \cdot 2 \cdot 1$

c. Write an expression involving factorials that could be used to find the coefficient of the third term of the expansion of $(m - n)^6$. (Do not actually calculate the coefficient.) $\frac{6!}{4!2!}$

Remember What You Learned

3. Without using Pascal's triangle or factorials, what is an easy way to remember the first two and last two coefficients for the terms of the binomial expansion of $(a + b)^n$? **Sample answer: The first and last coefficients are always 1. The second and next-to-last coefficients are always *n*, the power to which the binomial is being raised.**

Answers

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11-7 Study Guide and Intervention

The Binomial Theorem

Pascal's Triangle **Pascal's triangle** is the pattern of coefficients of powers of binomials displayed in triangular form. Each row begins and ends with 1 and each coefficient is the sum of the two coefficients above it in the previous row.

Pascal's Triangle	$(a+b)^0$	1
	$(a+b)^1$	1 1
	$(a+b)^2$	1 2 1
	$(a+b)^3$	1 3 3 1
	$(a+b)^4$	1 4 6 4 1
	$(a+b)^5$	1 5 10 10 5 1

Example **Use Pascal's triangle to find the number of possible sequences consisting of 3 *a*s and 2 *b*s.**

The coefficient 10 of the a^3b^2-term in the expansion of $(a+b)^5$ gives the number of sequences that result in three *a*s and two *b*s.

Exercises

Expand each power using Pascal's triangle.

1. $(a+5)^4$ $a^4 + 20a^3 + 150a^2 + 500a + 625$

2. $(x-2y)^6$ $x^6 - 12x^5y + 60x^4y^2 - 160x^3y^3 + 240x^2y^4 - 192xy^5 + 64y^6$

3. $(j-3k)^5$ $j^5 - 15j^4k + 90j^3k^2 - 270j^2k^3 + 405jk^4 - 243k^5$

4. $(2s+t)^7$ $128s^7 + 448s^6t + 672s^5t^2 + 560s^4t^3 + 280s^3t^4 + 84s^2t^5 + 14st^6 + t^7$

5. $(2p+3q)^6$ $64p^6 + 576p^5q + 2160p^4q^2 + 4320p^3q^3 + 4860p^2q^4 + 2916pq^5 + 729q^6$

6. $\left(a-\frac{b}{2}\right)^4$ $a^4 - 2a^3b + \frac{3}{2}a^2b^2 - \frac{1}{2}ab^3 + \frac{1}{16}b^4$

7. Ray tosses a coin 15 times. How many different sequences of tosses could result in 4 heads and 11 tails? **1365**

8. There are 9 true/false questions on a quiz. If twice as many of the statements are true as false, how many different sequences of true/false answers are possible? **84**

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11-7 Study Guide and Intervention *(continued)*

The Binomial Theorem

The Binomial Theorem

Binomial Theorem	If n is a nonnegative integer, then $(a+b)^n = 1a^nb^0 + \frac{n}{1}a^{n-1}b^1 + \frac{n(n-1)}{1\cdot 2}a^{n-2}b^2 + \frac{n(n-1)(n-2)}{1\cdot 2\cdot 3}a^{n-3}b^3 + \ldots + 1a^0b^n$

Another useful form of the Binomial Theorem uses **factorial** notation and sigma notation.

Factorial	If n is a positive integer, then $n! = n(n-1)(n-2)\cdot \ldots \cdot 2\cdot 1$.
Binomial Theorem, Factorial Form	$(a+b)^n = \frac{n!}{n!0!}a^nb^0 + \frac{n!}{(n-1)!1!}a^{n-1}b^1 + \frac{n!}{(n-2)!2!}a^{n-2}b^2 + \ldots + \frac{n!}{0!n!}a^0b^n$ $= \sum_{k=0}^{n}\frac{n!}{(n-k)!k!}a^{n-k}b^k$

Example 1 **Evaluate $\frac{11!}{8!}$.**

$$\frac{11!}{8!} = \frac{11\cdot 10\cdot 9\cdot 8\cdot 7\cdot 6\cdot 5\cdot 4\cdot 3\cdot 2\cdot 1}{8\cdot 7\cdot 6\cdot 5\cdot 4\cdot 3\cdot 2\cdot 1}$$
$$= 11\cdot 10\cdot 9 = 990$$

Example 2 **Expand $(a-3b)^4$.**

$$(a-3b)^4 = \sum_{k=0}^{4}\frac{4!}{(4-k)!k!}a^{4-k}(-3b)^k$$
$$= \frac{4!}{4!0!}a^4 + \frac{4!}{3!1!}a^3(-3b)^1 + \frac{4!}{2!2!}a^2(-3b)^2 + \frac{4!}{1!3!}a(-3b)^3 + \frac{4!}{0!4!}(-3b)^4$$
$$= a^4 - 12a^3b + 54a^2b^2 - 108ab^3 + 81b^4$$

Exercises

Evaluate each expression.

1. $5!$ **120** **2.** $\frac{9!}{7!2!}$ **36** **3.** $\frac{10!}{6!4!}$ **210**

Expand each power.

4. $(a-3)^6$ $a^6 - 18a^5 + 135a^4 - 540a^3 + 1215a^2 - 1458a + 729$

5. $(r+2s)^7$ $r^7 + 14r^6s + 84r^5s^2 + 280r^4s^3 + 560r^3s^4 + 672r^2s^5 + 448rs^6 + 128s^7$

6. $(4x+y)^4$ $256x^4 + 256x^3y + 96x^2y^2 + 16xy^3 + y^4$

7. $\left(2-\frac{m}{2}\right)^5$ $32 - 40m + 20m^2 - 5m^3 + \frac{5}{8}m^4 - \frac{1}{32}m^5$

Find the indicated term of each expansion.

8. third term of $(3x-y)^5$ $270x^3y^2$

9. fifth term of $(a+1)^7$ $35a^3$

10. fourth term of $(j+2k)^8$ $448j^5k^3$

11. sixth term of $(10-3t)^7$ $-510{,}300t^5$

12. second term of $\left(m+\frac{2}{3}\right)^9$ $6m^8$

13. seventh term of $(5x-2)^{11}$ $92{,}400{,}000x^5$

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11-7 Skills Practice

The Binomial Theorem

Evaluate each expression.

1. 8! 40,320
2. 10! 3,628,800
3. 12! 479,001,600
4. $\frac{15!}{13!}$ 210
5. $\frac{6!}{3!}$ 120
6. $\frac{10!}{2!8!}$ 45
7. $\frac{9!}{3!6!}$ 84
8. $\frac{20!}{15!5!}$ 15,504

Expand each power.

9. $(x - y)^3$
$x^3 - 3x^2y + 3xy^2 - y^3$
10. $(a + b)^5$
$a^5 + 5a^4b + 10a^3b^2 + 10a^2b^3 + 5ab^4 + b^5$
11. $(g - h)^4$
$g^4 - 4g^3h + 6g^2h^2 - 4gh^3 + h^4$
12. $(m + 1)^4$
$m^4 + 4m^3 + 6m^2 + 4m + 1$
13. $(r + 4)^3$
$r^3 + 12r^2 + 48r + 64$
14. $(a - 5)^4$
$a^4 - 20a^3 + 150a^2 - 500a + 625$
15. $(y - 7)^3$
$y^3 - 21y^2 + 147y - 343$
16. $(d + 2)^5$
$d^5 + 10d^4 + 40d^3 + 80d^2 + 80d + 32$
17. $(x - 1)^4$
$x^4 - 4x^3 + 6x^2 - 4x + 1$
18. $(2a + b)^4$
$16a^4 + 32a^3b + 24a^2b^2 + 8ab^3 + b^4$
19. $(c - 4d)^3$
$c^3 - 12c^2d + 48cd^2 - 64d^3$
20. $(2a + 3)^3$
$8a^3 + 36a^2 + 54a + 27$

Find the indicated term of each expansion.

21. fourth term of $(m + n)^{10}$ $120m^7n^3$
22. seventh term of $(x - y)^8$ $28x^2y^6$
23. third term of $(b + 6)^5$ $360b^3$
24. sixth term of $(s - 2)^9$ $-4032s^4$
25. fifth term of $(2a + 3)^6$ $4860a^2$
26. second term of $(3x - y)^7$ $-5103x^6y$

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11-7 Practice

The Binomial Theorem

Evaluate each expression.

1. 7! 5040
2. 11! 39,916,800
3. $\frac{9!}{5!}$ 3024
4. $\frac{20!}{18!}$ 380
5. $\frac{8!}{6!2!}$ 28
6. $\frac{8!}{5!3!}$ 56
7. $\frac{12!}{6!6!}$ 924
8. $\frac{41!}{3!38!}$ 10,660

Expand each power.

9. $(n + v)^5$ $n^5 + 5n^4v + 10n^3v^2 + 10n^2v^3 + 5nv^4 + v^5$
10. $(x - y)^4$ $x^4 - 4x^3y + 6x^2y^2 - 4xy^3 + y^4$
11. $(x + y)^6$ $x^6 + 6x^5y + 15x^4y^2 + 20x^3y^3 + 15x^2y^4 + 6xy^5 + y^6$
12. $(r + 3)^5$ $r^5 + 15r^4 + 90r^3 + 270r^2 + 405r + 243$
13. $(m - 5)^5$ $m^5 - 25m^4 + 250m^3 - 1250m^2 + 3125m - 3125$
14. $(x + 4)^4$ $x^4 + 16x^3 + 96x^2 + 256x + 256$
15. $(3x + y)^4$ $81x^4 + 108x^3y + 54x^2y^2 + 12xy^3 + y^4$
16. $(2m - y)^4$ $16m^4 - 32m^3y + 24m^2y^2 - 8my^3 + y^4$
17. $(w - 3z)^3$ $w^3 - 9w^2z + 27wz^2 - 27z^3$
18. $(2d + 3)^6$ $64d^6 + 576d^5 + 2160d^4 + 4320d^3 + 4860d^2 + 2916d + 729$
19. $(x + 2y)^5$ $x^5 + 10x^4y + 40x^3y^2 + 80x^2y^3 + 80xy^4 + 32y^5$
20. $(2x - y)^5$ $32x^5 - 80x^4y + 80x^3y^2 - 40x^2y^3 + 10xy^4 - y^5$
21. $(a - 3b)^4$ $a^4 - 12a^3b + 54a^2b^2 - 108ab^3 + 81b^4$
22. $(3 - 2z)^4$ $16z^4 - 96z^3 + 216z^2 - 216z + 81$
23. $(3m - 4n)^3$ $27m^3 - 108m^2n + 144mn^2 - 64n^3$
24. $(5x - 2y)^4$ $625x^4 - 1000x^3y + 600x^2y^2 - 160xy^3 + 16y^4$

Find the indicated term of each expansion.

25. seventh term of $(a + b)^{10}$ $210a^4b^6$
26. sixth term of $(m - n)^{10}$ $-252m^5n^5$
27. ninth term of $(r - s)^{14}$ $3003r^6s^8$
28. tenth term of $(2x + y)^{12}$ $1760x^3y^9$
29. fourth term of $(x - 3y)^6$ $-540x^3y^3$
30. fifth term of $(2x - 1)^9$ $4032x^5$

31. **GEOMETRY** How many line segments can be drawn between ten points, no three of which are collinear, if you use exactly two of the ten points to draw each segment? 45

32. **PROBABILITY** If you toss a coin 4 times, how many different sequences of tosses will give exactly 3 heads and 1 tail or exactly 1 head and 3 tails? 8